

On pseudo-finite model theory

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<p>A model in mathematic logic is called pseudo-finite, in case it satisfies only such sentences of first-order predicate logic that have a finite model. Its main part modelled based on Jouko Väänänen's article "Pseudo-finite model theory", this text studies classic model theory restricted to pseudo-finite models. We provide a range of classic results expressed in pseudo-finite terms, while showing that a set of other well-known theorems fail when restricted to the pseudo-finite, unless modified substantially. The main finding remains that a major portion of the classic theory, including Compactness Theorem, Craig Interpolation Theorem and Lidström Theorem, holds in an analogical form in the pseudo-finite theory.</p> <p>The thesis begins by introducing the basic first-order model theory with the restriction to relational formulas. This purely technically motivated limitation doesn't exclude any substantial results or methods of the first-order theory, but it simplifies many of the proofs. The introduction behind, the text moves on to present all the classic results that will later on be studied in terms of the pseudo-finite. To enable and ease this, we also provide some powerful tools, such as Ehrenfeucht-Fraïssé games.</p> <p>In the main part of the thesis we define pseudo-finiteness accurately and build a pseudo-finite model theory. We begin from easily adaptable results such as Compactness and Löwenheim-Skolem Theorems and move on to trickier ones, exemplified by Craig Interpolation and Beth Definability. The section culminates to a Lidström Theorem, which is easy to formulate but hard to prove in pseudo-finite terms.</p> <p>The final chapter has two independent sections. The first one studies the requirements of a sentence for having a finite model, illustrates a construction of a finite model for a sentence that has one, and culminates into an exact finite model existence theorem. In the second one we define a class of models with a certain, island-like structure. We prove that the elements of this class are always pseudo-finite, and at the very end the text, we present a few examples of this class.</p>			
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1 Introduction

This text is about the theory of models which satisfy only first-order sentences that have a finite model. We call these models pseudo-finite. After some basics in relational first-order model theory, we will show that many of the classic results in the subject can be formulated in terms of pseudo-finite models, primarily following [1]. We also look into what is required from a sentence to be true only in finite models and provide a class of models that are always pseudo-finite.

2 First-order model theory

2.1 First-order logic

In this section we provide the basic definitions of relational first-order logic.

Definition 2.1. An alphabet L is any set of constant, relation and function symbols, such that each relation symbol R and each function symbol f have an arity, $\#R, \#f \in \mathbb{N} \setminus \{0\}$.

When not mentioned otherwise, we let $L = \{R_i, f_j, c_k | i \in I, j \in J, k \in K\}$ be a fixed but arbitrary alphabet. So whenever we refer to e.g. a function symbol f without specifying an alphabet, then by default, $f = f_j \in L$ for some $j \in J$.

Definition 2.2. L -terms are defined as follows:

- (i) Variable symbols v_i , $i \in \mathbb{N}$, are L -terms
- (iii) Constant symbols $c_i \in L$ are L -terms
- (iv) If $f \in L$ is a function symbol, $n = \#f$ and $t_1 \dots, t_n$ are L -terms, then $f(t_1 \dots, t_n)$ is an L -term.

Definition 2.3. L -atomic formulas are defined as follows:

- (i) If t and u are L -terms, then $t = u$ is an L -atomic formula
- (ii) If $R \in L$ is a relation symbol, $\#R = n$ and $t_1 \dots, t_n$ are L -terms, then $R(t_1 \dots, t_n)$ is an L -atomic formula.

Definition 2.4. L -formulas are defined as follows:

- (i) L -atomic formulas are L -formulas
- (iii) if ϕ is an L -formula, then $\neg\phi$ is an L -formula called the negation of ϕ
- (iv) if ϕ and ψ are L -formulas, then $(\phi \wedge \psi)$ is an L -formula called the conjunction of ϕ and ψ
- (v) if ϕ is an L -formula and $i \in \mathbb{N}$, then $\exists v_i \phi$ is an L -formula. We say ϕ is a kvantified formula and call \exists the existential quantifier.

These formulas are also called the first-order formulas. We will often refer to symbols, terms and formulas without specifying an alphabet they belong to, but we then assume they are L -symbols, L -terms and L -formulas for the fixed but arbitrary alphabet L noted earlier.

If there is no risk of confusion, we write $\phi \wedge \psi$ rather than $(\phi \wedge \psi)$. The following notations will be used:

$$\begin{aligned}\phi \vee \psi &:= \neg(\neg\phi \wedge \neg\psi) \\ \phi \rightarrow \psi &:= \neg\phi \vee \psi \\ \phi \leftrightarrow \psi &:= (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi) \\ \forall v_i \phi &:= \neg \exists v_i \neg\phi\end{aligned}$$

We call \forall the universal quantifier.

Throughout the text, we shall use " $:=$ " when defining notations to highlight what is the new notation and what is its definition; if we write $\varphi := \psi$ or $\psi =: \varphi$, we mean that φ is a notation for ψ .

We shall drop the brackets from $(\phi \wedge \psi)$, in case there is no essential risk of confusion. Also, we use letters x, y, z and their abbreviations to denote arbitrary variable symbols.

Further, we sometimes use the notation $\phi \wedge \psi$ such that ϕ is a formula but ψ is either a formula or the empty set. In the latter case, we define $\phi \wedge \psi := \psi$.

For a larger list of formulas, $\phi_1 \dots, \phi_n$, we use the notation $\bigwedge_{i \in \{1, \dots, n\}} \phi_i$ when we mean $\phi_1 \wedge \dots \wedge \phi_n$, and $\bigvee_{i \in \{1, \dots, n\}} \phi_i$ when we mean $\phi_1 \vee \dots \vee \phi_n$.

Definition 2.5. Given a variable symbol v_i and a formula ϕ , we define the notion " v_i is free in ϕ " as follows:

- (i) ϕ is atomic: v_i is free in ϕ , if v_i appears in ϕ
- (ii) $\phi = \neg\psi$ for some formula ψ : v_i is free in ϕ , if it is free in ψ
- (iii) $\phi = \psi \wedge \theta$ for some formulas ψ and θ : v_i is free in ϕ , if it is free in ψ or in θ
- (iv) $\phi = \exists v_j \psi$ for some formula ψ : v_i is free in ϕ , if it is free in ψ and $i \neq j$.

If v_i was free in ϕ , then we say that $\exists v_i$ bounds v_i in $\exists v_i \phi$. Further, if v_i is not free in ϕ , then we say that v_i is bounded in ϕ .

Definition 2.6. By sentence we mean a formula in which no variable symbol is free. By theory we mean a class of sentences, and if T is a theory all the sentences in which are L -sentences, then we say that T is an L -theory.

Unless otherwise stated, when presenting a list of variable symbols by a list mutually distinct of symbols that have no fixed definition, we assume that each of the latter symbols depicher a unique variable symbol. Thus, if we use $x = (x_1, \dots, x_n)$ to represent a list of fixed but arbitrary variable symbols, we suppose that when $i \neq j$, x_i and x_j represent different variable symbols. By $x_i = x_j$ we mean that the variable symbols x_i and x_j represent are the same, and by $x_i \neq x_j$ we respectively mean that x_i and x_j represent distinct variable symbols. Further, instead of stating that "let $x = (x_1, \dots, x_n)$ represent a list of fixed but arbitrary variable symbols" we usually just say "let $x = (x_1, \dots, x_n)$ be variables" or "let $x = (x_1, \dots, x_n)$ ".

Now if $x = (x_1, \dots, x_n)$ are variables and ϕ a formula, then by $\phi(x)$ and $\phi(x_1, \dots, x_n)$ we mean that x_1, \dots, x_n are presicely the free variable symbols of ϕ . In addition, given a variable y and variable sequences $z = (z_1, \dots, z_k)$, $z^* = (z_{k+1}, \dots, z_n)$, $1 \leq k \leq n$, by $\phi(x_1, \dots, x_{m-1}, y, x_{m+1}, \dots, x_n)$ we mean the formula obtained from $\phi(x_1, \dots, x_n)$ by replacing x_m with y , and by $\phi(z, z^*)$ we mean the formula obtained from $\phi(x_1, \dots, x_n)$ by replacing x_1, \dots, x_k with z_1, \dots, z_k and x_{k+1}, \dots, x_n with z_{k+1}, \dots, z_n .

Definition 2.7. A model is a sequence

$$\mathcal{M} = (M, (R_i^{\mathcal{M}})_{i \in I}, (f_j^{\mathcal{M}})_{j \in J}, (c_k^{\mathcal{M}})_{k \in K}),$$

where

(i) M is a non-empty set, called the domain of the model \mathcal{M} and donated also by $\text{dom}(\mathcal{M})$. This set is in addition called the universe of \mathcal{M} . When there is no risk of confusion, we shall donate the universums of models $\mathcal{M}, \mathcal{N}, \mathcal{A}, \mathcal{B}, \dots$ as M, N, A, B, \dots without writing out that $\mathcal{M} = (M, (R_i^{\mathcal{M}})_{i \in I}, (f_j^{\mathcal{M}})_{j \in J}, (c_k^{\mathcal{M}})_{k \in K})$, $\mathcal{N} = (N, (R_i^{\mathcal{N}})_{i \in I}, (f_j^{\mathcal{N}})_{j \in J}, (c_k^{\mathcal{N}})_{k \in K})$ and so on). The elements of the domain of a model are called the elements or the points of the model.

- (ii) $R_i^{\mathcal{M}} \subseteq M^{\#R}$
 - (iii) $f_j^{\mathcal{M}} : M^{\#R} \rightarrow M$
 - (iv) $c_k^{\mathcal{M}} \in M$
- for all $i \in I, j \in J, k \in K$.

The $R_i^{\mathcal{M}}$, $f_j^{\mathcal{M}}$ and $c_k^{\mathcal{M}}$ are called the interpretations of R_i , f_j and c_k . If all the symbols R_i , f_j and c_k belong to an alphabet L , then we say \mathcal{M} is an L -model.

Definition 2.8. Let \mathcal{M} be an L -model. Then

$$s : L \cup \{v_i : i \in \mathbb{N}\} \mapsto M \cup \bigcup \{c_i^{\mathcal{M}} : i \in I\} \cup \bigcup \{f_j^{\mathcal{M}} : j \in J\} \cup \bigcup \{R_k^{\mathcal{M}} : k \in K\}$$

is an interpretation function in \mathcal{M} , if

- (i) $s(v_i) \in M$ for all variable symbols v_i , $i \in \mathbb{N}$
- (ii) $s(c_i) = c_i^{\mathcal{M}}$ for all $i \in I$
- (iii) $s(f_j) = f_j^{\mathcal{M}}$ for all $j \in J$
- (iv) $s(R_k) = R_k^{\mathcal{M}}$ for all $k \in K$.

If s is an interpretation function and $a \in M$, $i \in \mathbb{N}$, then by $s(i/a)$ we mean the interpretation formula that is otherwise identical to s but $s(v_i) = a$.

If f is an n -ary L -function symbol, then by $s(f(x_1, \dots, x_n))$ we mean $s(f)(s(x_1), \dots, s(x_n))$. Thus we have defined the notion $s(t)$ for any term.

Definition 2.9. The Tarski Truth Definition

Suppose $\phi(x_1, \dots, x_n)$ is an L -formula, \mathcal{M} is an L -model, $a = (a_1, \dots, a_n) \in M^n$ and s is such an interpretation function that $s(x_i) = a_i$ for all $i \in \{1, \dots, n\}$. Then we define $\mathcal{M} \models_s \phi$ as follows:

- (i) ϕ is $t = u$: $\mathcal{M} \models_s \phi$ iff $s(t) = s(u)$
- (ii) $\phi = R_k(t_1, \dots, t_m)$: $\mathcal{M} \models_s \phi$ iff $(s(t_1), \dots, s(t_m)) \in R_k^{\mathcal{M}}$
- (iii) $\phi = \neg\psi$: $\mathcal{M} \models_s \phi$ iff $\mathcal{M} \not\models_s \psi$
- (iv) $\phi = \psi(x) \wedge \theta(x)$: $\mathcal{M} \models_s \phi$ iff $\mathcal{M} \models_s \psi$ and $\mathcal{M} \models_s \theta$
- (v) $\phi = \exists v_i \psi(x, v_i)$: $\mathcal{M} \models_s \phi$ iff for some $a \in M$, $\mathcal{M} \models_{s(i/a)} \psi$.

If $\mathcal{M} \models_s \phi$, then we say ϕ is true in \mathcal{M} under the interpretation s .

Remark 2.10. Suppose \mathcal{M} is an L -model, $\phi(x_1, \dots, x_n)$ is an L -formula and s and s' are interpretation functions in \mathcal{M} such that $s' \upharpoonright \{x_1, \dots, x_n\} = s \upharpoonright \{x_1, \dots, x_n\}$. Then $\mathcal{M} \models_{s'} \phi$ if and only if $\mathcal{M} \models_s \phi$.

By remark 1.10 we can make a few simplifications in our notation and terminology. Suppose the assumptions of the remark hold, and suppose $a \in M^n$ and $s(x_i) = a_i$ for all $i \in \{1, \dots, n\}$. In this case we shall normally use the notation $\mathcal{M} \models \phi(a_1, \dots, a_n)$, or shortly, $\mathcal{M} \models \phi(a)$, when we mean $\mathcal{M} \models_s \phi$. (We in fact do not assume that x_1, \dots, x_n are free in ϕ when writing $\mathcal{M} \models \phi(a_1, \dots, a_n)$, even though we do make that assumption when notating $\phi(x_1, \dots, x_n)$.) Further, if φ is an L -sentence, then $\mathcal{M} \models_s \varphi$ either for all or for no interpretation function s , so we may simply notate $\mathcal{M} \models \varphi$.

If ϕ is a sentence and $\mathcal{M} \models \phi$, then we say that ϕ is true in \mathcal{M} and that \mathcal{M} satisfies ϕ , and also that ϕ has a model. For a theory T , by $\mathcal{M} \models T$ we mean that $\mathcal{M} \models \phi$ for every $\phi \in T$. If $\mathcal{M} \models T$, we say \mathcal{M} satisfies T .

For an alphabet L , let $Str(L)$ denote the class of all L -models and $Fo(L)$ the class of all L -sentences. Then the triple $L_{\omega, \omega} = (Str(L), Fo(L), \models)$, where $\models \subseteq Str(L) \times Fo(L)$ is the truth relation defined above, is called the relational first-order logic. For the sake of simplicity, we shall refer to it as simply the first order logic from here on.

Definition 2.11. For a formula ϕ and for terms t and u , $\psi(t/u)$ is ψ with every appearance of the term t replaced by u .

Definition 2.12. Suppose T is an L -theory and ϕ an L -formula. Then by

$$T \vdash \phi$$

we mean that for any L -model \mathcal{M} and interpretation function s , $\mathcal{M} \models T$ implies $\mathcal{M} \models_s \phi$. If $T = \emptyset$ and all L -models and their interpretation functions satisfy ϕ , we write

$$\vdash \phi.$$

Definition 2.13. If ϕ and ψ are formulas and $\vdash \phi \leftrightarrow \psi$, then we say that ϕ and ψ are equivalent and that ϕ is equivalent to ψ .

Definition 2.14. An L -theory T is

(i) consistent if there is a model \mathcal{M} such that $\mathcal{M} \models T$ (i.e. T has a model). Otherwise T is inconsistent.

(ii) complete if for every L -sentence ϕ , $T \vdash \phi$ or $T \vdash \neg\phi$.

Definition 2.15. (i) Let \mathcal{M} and \mathcal{N} be L -models. If $dom(\mathcal{N}) \subseteq dom(\mathcal{M})$, $R_i^{\mathcal{N}} = R_i^{\mathcal{M}} \cap dom(\mathcal{N})$, $f_j^{\mathcal{N}} \subseteq f_j^{\mathcal{M}}$ and $c_k^{\mathcal{N}} = c_k^{\mathcal{M}}$ for all $i \in I, j \in J, k \in K$, then we say that \mathcal{N} is a submodel of \mathcal{M} and write $\mathcal{N} \subseteq \mathcal{M}$.

(ii) For $A \subseteq M$,

$$\mathcal{M} \upharpoonright A = (A, (c_i^{\mathcal{M}})_{i \in I}, (f_j^{\mathcal{M}} \upharpoonright A)_{j \in J}, (R_k^{\mathcal{M}} \upharpoonright A)_{k \in K}),$$

which by definition 2.7 is a model if $f_j^{\mathcal{N}}(a) \in A$ for all $a \in A, i \in I$. Similarly, if $\emptyset \neq \tau \subseteq L$, then

$$\mathcal{M} \upharpoonright \tau = (M, (c_i^{\mathcal{M}})_{i \in \{i: c_i \in \tau\}}, (f_j^{\mathcal{M}})_{j \in \{j: f_j \in \tau\}}, (R_k^{\mathcal{M}})_{k \in \{k: R_k \in \tau\}}),$$

which always is a model by definition 2.7, and

$$\mathcal{M} \upharpoonright (A, \tau) = (A, (c_i^{\mathcal{M}})_{i \in \{i: c_i \in \tau\}}, (f_j^{\mathcal{M}} \upharpoonright A)_{j \in \{j: f_j \in \tau\}}, (R_k^{\mathcal{M}} \upharpoonright A)_{k \in \{k: R_k \in \tau\}}),$$

which is a model if $f_j^{\mathcal{N}}(a) \in A$ for all $a \in A, j \in \{j : f_j \in \tau\}$.

Definition 2.16. (i) An atomic formula ϕ is relational if it is of one of the following forms: $R(x_1, \dots, x_n), f(x_1, \dots, x_n) = y, y = f(x_1, \dots, x_n), c_i = x$ or $x = c_i$.

(ii) Relational atomic formulas are relational formulas.

(iii) If ϕ and ψ are relational formulas, then also $\neg\phi, \phi \wedge \psi$ and $\exists\phi$ are relational.

Definition 2.17. (i) Let t be a term. The relationality rank of t , $rr(t)$, is defined as follows: If $t = v_i$ then $rr(t) = 0$, if $t = c_i$ then $rr(t) = 1$, and if $t = f(u_1, \dots, u_n)$ then $rr(t) = 1 + \max\{rr(u_1), \dots, rr(u_n)\}$.

(ii) Suppose ϕ is an atomic formula. Its relationality rank $rr(\phi)$ is defined as follows: If $\phi = R(t_1, \dots, t_n)$, then $rr(\phi) = \max\{rr(t_1), \dots, rr(t_n)\}$, and if $\phi = t = u$, then $rr(\phi) = rr(t) + rr(u) - 1$.

We remark that an atomic formula ϕ is relational, if and only if $rr(\phi) \leq 0$.

Lemma 2.18. *For any atomic formula $\phi(x), x = (x_1, \dots, x_n)$ there is a relational formula $\psi(x)$ equivalent to $\phi(x)$.*

Proof. By induction on $rr(\phi)$. If $rr(\phi) \leq 0$, $\phi(x)$ is relational itself. Suppose $rr(\phi) = p + 1$. We show the claim for $\phi = R(t_1, \dots, t_n); c_i = c_j, f(t_1, \dots, t_n) = u$ and $u = f(t_1, \dots, t_n)$ satisfy the claim by an analogous argument. Now ϕ is equivalent with $\psi^* := \exists y_1, \dots, \exists y_1 (R(y_1, \dots, y_n) \wedge \bigwedge_{1 \leq i \leq n} y_i = t_i)$, where $rr(y_i = t_i) \leq p$ for all $1 \leq i \leq n$ and hence by the induction assumption, there are relational formulas θ_i equivalent to $y_i = t_i$ for all $1 \leq i \leq n$. Thus ψ^* along with ϕ are equivalent to a relational formula. \square

Lemma 2.19. *For any formula $\phi(x), x = (x_1, \dots, x_n)$ there is a relational formula $\psi(x)$ equivalent to $\phi(x)$.*

Proof. By induction on ϕ . For an atomic ϕ the claim follows from lemma 2.18. If $\phi = \neg\phi$ or $\phi = \psi \wedge \theta$, the claim follows trivially from the induction assumption. If $\phi = \exists x\psi$, then by the induction assumption, ψ is equivalent to a relational θ . Thus $\vdash \phi \leftrightarrow \exists x\theta$, where $\exists x\theta$ is relational. \square

By lemma 2.19, we can make the following restriction: From here on, we assume the logic used, $L_{\omega, \omega}$, is relational, i.e. whenever referring to a formula, we assume the formula is relational.

Definition 2.20. The quantifier rank $qr(\phi)$ of a formula ϕ is defined as follows:

- (i) ϕ is atomic: $qr(\phi) = 0$
- (ii) $qr(\neg\psi) = qr(\psi)$
- (iii) $qr(\psi \wedge \theta) = \max\{qr(\psi), qr(\theta)\}$
- (iv) $qr(\exists x\psi) = qr(\psi) + 1$.

If $qr(\phi) = 0$ we say that ϕ is unquantified.

Lemma 2.21. Suppose L is finite. Then for all $k, n \in \mathbb{N}$ there is a finite set F_n^k of L -formulas of quantifier rank $\leq k$ such that

- (i) if $\phi \in F_n^k$, then no other variables than v_0, \dots, v_n appear free in ϕ
- (ii) for any L -formula ψ , $qr(\psi) \leq k$, in which no other variables than v_0, \dots, v_n appear free, there is an equivalent formula ϕ such that $\phi \in F_n^k$.

Proof. By induction on $k \in \mathbb{N}$.

$k = 0$: Let $\theta_1, \dots, \theta_n$ list all the atomic L -formulas in which no other variables than v_0, \dots, v_n appear. Let Φ_n be the finite set of the formulas $\bigwedge_{i \in X} \theta_i \wedge \bigwedge_{i \in Y} \neg \theta_i$ for any $X, Y \subseteq \{1, \dots, n\}$. Then let ψ_i , $1 \leq i \leq m$, enumerate Φ_n and define

$$F_n^0 = \left\{ \bigvee_{i \in X} \psi_i : X \subseteq \{1, \dots, n\} \right\}.$$

Since for any formulas ϕ and ψ , $\neg\neg\phi$ and $\phi \wedge \phi$ are equivalent to ϕ and $\neg(\phi \wedge \psi)$ is equivalent to $\neg\phi \vee \neg\psi$, any unquantified formula θ can be equivalently expressed in the form $\sigma := \bigvee \{\bigwedge_{1 \leq i \leq k} \phi_k\}$, where every ϕ_k is an atomic or a negated atomic formula. Now by the definition of the sets F_n^0 , there is $n \in \mathbb{N}$ such that $\sigma \in F_n^0$.

$k = p + 1$: Define

$$\Psi_n^k = F_n^p \cup \{\exists v_{n+1}\phi, \neg\exists v_{n+1}\phi : \phi \in F_{n+1}^p\}.$$

Let θ_i , $1 \leq i \leq k$, list the formulas of Ψ_n^k and let Φ_n^k be the finite set of the formulas $\bigwedge_{i \in X} \theta_i$ for any $X \subseteq \{1, \dots, k\}$. Then let ψ_i , $1 \leq i \leq m$, enumerate Φ_n^k and define

$$F_n^k = F_n \cap \left\{ \bigvee_{i \in Y} \psi_i : Y \subseteq \{1, \dots, m\} \right\},$$

where F_n is the set of L -formulas that no other variables than v_0, \dots, v_n appear free in. Again, for any formulas ϕ and ψ , $\neg\neg\phi$ and $\phi \wedge \phi$ are equivalent to ϕ and $\neg(\phi \wedge \psi)$ is equivalent to $\neg\phi \vee \neg\psi$, so any formula θ , $qr(\theta) \leq k$, in which no other variables than v_0, \dots, v_n appear free, can be equivalently expressed in the form $\sigma := \bigvee \{\bigwedge_{1 \leq i \leq k} \phi_k\}$, where every ϕ_k either is an element of F_n^p , or $\phi = \{\exists v_{i_k} \psi(v_{i_1}, \dots, v_{i_k}), \neg \exists v_{i_k} \psi(v_{i_1}, \dots, v_{i_k})\}$, where $i_k > i_j$ for all $j < k$ and $\psi \in F_n^p$. Hence by the definition of the sets F_n^k , $\sigma \in F_n^k$. \square

2.2 On partial isomorphisms and their applications

Definition 2.22. (i) Suppose \mathcal{A} and \mathcal{B} are models, $X \subseteq A$ and $f : X \rightarrow \mathcal{B}$. Then f is a partial isomorphism from \mathcal{A} to \mathcal{B} if for all atomic formulas $\phi(x_1, \dots, x_n)$ and $a \in X^n$, $\mathcal{A} \models \phi(a)$ if and only if $\mathcal{B} \models \phi(f(a))$.

(ii) If $f : \mathcal{A} \rightarrow \mathcal{B}$ is a partial isomorphism and $\mathcal{A} = \mathcal{B}$, then f is a partial automorphism.

(iii) If $X = A$, then f is an isomorphism, and if $\mathcal{A} = \mathcal{B}$, then f is an automorphism.

(iv) If there exists an isomorphism between \mathcal{A} and \mathcal{B} , we say that \mathcal{A} and \mathcal{B} are isomorphic.

Accurately, a partial isomorphism f from \mathcal{A} to \mathcal{B} is a function $f : \text{dom}(\mathcal{A}) \rightarrow \text{dom}(\mathcal{B})$. However, we use the notation $f : X \rightarrow \mathcal{B}$, $X \subseteq \text{dom}(\mathcal{A})$ to highlight that f is a partial isomorphism in relation to models \mathcal{A} and \mathcal{B} .

Definition 2.23. Suppose $U \in L$ is a unary relation symbol and $\psi(x, z)$, $(x, z) = (x_1, \dots, x_n, z_1, \dots, z_m)$, and θ are L -formulas, such that none of the variable symbols $x_1, \dots, x_n, z_1, \dots, z_m$ appear in θ . Let \mathcal{A} be an L -model and let $b \in A$. Then formulas $\theta^{(U)}$ and $\theta^{(\psi(b, z))}$ are defined as follows:

- (i) θ is atomic: $\theta^{(U)}, \theta^{(\psi(b, z))} = \theta$
- (ii) $\theta = \neg\phi$: $\theta^{(U)} = \neg\phi^{(U)}$, $\theta^{(\psi(b, z))} = \neg\phi^{(\psi(b, z))}$
- (iii) $\theta = \phi \wedge \varphi$: $\theta^{(U)} = \phi^{(U)} \wedge \varphi^{(U)}$, $\theta^{(\psi(b, z))} = \phi^{(\psi(b, z))} \wedge \varphi^{(\psi(b, z))}$
- (iv) $\theta = \exists y\phi$: $\theta^{(U)} = \exists y(U(y) \wedge \phi^{(U)})$ and $\theta^{(\psi(b, z))} = \exists y(\psi((b, y)) \wedge \phi^{(\psi(b, z))})$

If z is a single variable symbol and there is no risk of confusion, we mark $\theta^{(\psi(b, z))} =: \theta^{(\psi(b))}$ and $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b, z) =: \mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b)$.

Lemma 2.24. Suppose $U \in L$ is a unary relation symbol and $\psi(x, z)$, $x = (x_1, \dots, x_n)$, and $\theta(y)$, $y = (y_1, \dots, y_k)$, are L -formulas. Let \mathcal{A} be an L -model and $b \in A^n$ such that $\mathcal{A} \upharpoonright U^{\mathcal{A}}$ and $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b, z)$ are submodels of \mathcal{A} . Then for all $a \in (U^{\mathcal{A}})^k$, $\mathcal{A} \models \theta^{(U)}(a)$ if and only if $\mathcal{A} \upharpoonright U^{\mathcal{A}} \models \theta(a)$, and for all $a \in (\psi^{\mathcal{A}}(b, z))^k$, $\mathcal{A} \models \theta^{(\psi(b))}(a)$ if and only if $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \theta(a)$.

Proof. By induction on ψ . $\mathcal{A} \models \theta^{(U)}(a)$ is a special case of $\mathcal{A} \models \theta^{(\psi(b))}(a)$, so it suffices to prove that $\mathcal{A} \models \theta^{(\psi(b))}(a)$ if and only if $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \theta(a)$.

- (i) θ is atomic: $\mathcal{A} \models \theta^{(\psi(b))}(a)$ iff $\mathcal{A} \models \theta(a)$ iff $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \theta(a)$ since $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(z, b)$ is a submodel of \mathcal{A} and $a \in (\psi^{\mathcal{A}}(z, b))^k$.
- (ii) $\theta = \neg\phi$: $\mathcal{A} \models \theta^{(\psi(b))}(a)$ iff $\mathcal{A} \not\models \phi^{(\psi(b))}(a)$ iff $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \not\models \phi(a)$ iff $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \theta(a)$.
- (iii) $\theta = \phi \wedge \varphi$: $\mathcal{A} \models \theta^{(\psi(b))}(a)$ iff $\mathcal{A} \models \phi^{(\psi(b))}(a)$ and $\mathcal{A} \models \varphi^{(\psi(b))}(a)$ iff $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \phi(a)$ and $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \varphi(a)$ iff $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \theta(a)$.
- (iv) $\theta = \exists y\phi$, $y \neq z$: $\mathcal{A} \models \theta^{(\psi(b, z))}(a)$ iff there is $c \in A$ such that $\mathcal{A} \models \psi(b, c) \wedge \phi^{(\psi(b, z))}(a, c)$ iff there is $c \in \psi^{\mathcal{A}}(b, z)$ such that $\mathcal{A} \models \phi^{(\psi(b, z))}(a, c)$ iff there is $c \in \psi^{\mathcal{A}}(b, z)$ such that $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \phi(a, c)$ iff $\mathcal{A} \upharpoonright \psi^{\mathcal{A}}(b) \models \theta(a)$.

□

Definition 2.25. Suppose $\mathcal{A} \subseteq \mathcal{B}$ are L -models. Then we write

- (i) $\mathcal{A} \preceq \mathcal{B}$ if for all L -formulas $\psi(x_1, \dots, x_n)$ and all $a \in A^n$, $\mathcal{A} \models \psi(a)$ if and only if $\mathcal{B} \models \psi(a)$
- (ii) $\mathcal{A} \preceq_k \mathcal{B}$ if for all L -formulas $\psi(x_1, \dots, x_n)$ with $qr(\psi) \leq k$ and for all $a \in A^n$, $\mathcal{A} \models \psi(a)$ if and only if $\mathcal{B} \models \psi(a)$
- (iii) $\mathcal{A} \equiv \mathcal{B}$ if \mathcal{A} and \mathcal{B} satisfy precisely the same L -sentences; in this case we say that \mathcal{A} and \mathcal{B} are elementary equivalent
- (iv) $\mathcal{A} \equiv_k \mathcal{B}$ if for all L -sentences ψ with $qr(\psi) \leq k$, $\mathcal{A} \models \psi$ if and only if $\mathcal{B} \models \psi$

Lemma 2.26. *Tarski-Vaught for formulas with a limited quantifier rank.*

$\mathcal{A} \preceq_k \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all formulas $\psi(x_1, \dots, x_n, y)$ with $qr(\psi) < k$ and for all $a \in A^n$ the following holds: if $\mathcal{B} \models \exists y \psi(a, y)$, then there is $b \in A$ such that $\mathcal{B} \models \psi(a, b)$.

Proof. Suppose $k \in \mathbb{N}$. Let us prove by induction on the structure of an L -formula $\varphi(z_1, \dots, z_n)$ that if $qr(\varphi) \leq k$ and $b \in A^n$, then $\mathcal{A} \models \varphi(b)$, if and only if $\mathcal{B} \models \varphi(b)$.

1) φ is atomic: Directly from the assumption $\mathcal{A} \subseteq \mathcal{B}$.

2) $\varphi = \neg\psi$ or $\varphi = \psi \wedge \theta$: Considering that $qr(\psi), qr(\theta) \leq qr(\varphi)$, the induction claim follows from the induction assumption.

3) $\varphi = \exists y_i \psi$: Two directions:

" \Rightarrow ": If $\mathcal{A} \models \varphi(b)$, then there is $c \in A$ such that $\mathcal{A} \models \psi(b, c)$. Since $qr(\psi) \leq k$, the induction assumption yields $\mathcal{B} \models \psi(b, c)$, in which case $\mathcal{B} \models \varphi(b)$.

" \Leftarrow ": If $\mathcal{B} \models \varphi(b)$, then the assumption provides $c \in A$ such that $\mathcal{B} \models \psi(a, b)$. Since $qr(\psi) \leq k$, the induction assumption yields $\mathcal{A} \models \psi(a, b)$, so $\mathcal{A} \models \varphi(b)$.

□

Corollary 2.27. $\mathcal{A} \preceq_k \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and there is $k \in \mathbb{N} \setminus \{0\}$ such that for all formulas $\psi(x, y)$, $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_k)$, with $qr(\psi) < k$ and for all $a \in A^n$ the following holds: if $\mathcal{B} \models \exists y_1 \dots \exists y_k \psi(a, y_1, \dots, y_k)$, then there is $b \in A^k$ such that $\mathcal{B} \models \psi(a, b)$.

Proof. Suppose $\mathcal{A} \subseteq \mathcal{B}$ and $\psi(x_1, \dots, x_n, y)$ is an L -formula with $qr(\psi) < k$. Suppose $a \in A^n$ and $\mathcal{B} \models \exists y \psi(a, y)$. Then $\mathcal{B} \models \exists y_1 \dots \exists y_k \phi(a, y_1, \dots, y_k)$, where

$$\phi = \psi \wedge \bigwedge_{0 \leq i \leq k} y_i = y_i.$$

Now there is $b = (b_1, \dots, b_n) \in A^k$ such that $\mathcal{B} \models \phi(a, b)$, which implies $\mathcal{B} \models \psi(a, b_1)$. Thus lemma 2.26 yields $\mathcal{A} \preceq_k \mathcal{B}$.

□

Definition 2.28. Similiar to [2, definition 7.6].

Suppose \mathcal{A} and \mathcal{B} are L -models, $a \in A^n$, $b \in B^n$ and $A \cap B = \emptyset$.

(i) An Ehrenfeucht-Fraïssé game of lenght $k \in \mathbb{N}$, denoted by $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$, is a game played by two players, I and II . For $k \geq 1$, each round $1 \leq m < k$ begins by player I choosing $c_m \in A \cup B$, which is followed by player II choosing a partial isomorphism $f_m : A \rightarrow B$ such that $c_m \in \text{dom}(f_m) \cup \text{rng}(f_m)$ and $f_m \upharpoonright \text{dom}(f_{m-1}) = f_{m-1}$, where $f_0 = \{(a_i, b_i) : 1 \leq i \leq n, (a_1, \dots, a_n) = a, (b_1, \dots, b_n) = b\}$. For any $k \in \mathbb{N}$, player II wins the game if every f_m , $m \leq k$, is a partial isomorphism; otherwise I wins the game.

(ii) A function sequence $(g_i)_{1 \leq i \leq k}$ is a strategy for player II in $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$, if for all $1 \leq i \leq k$, g_i is a function from $(A \cup B)^i$ to the partial maps $A \rightarrow B$. We call $(g_i)_{1 \leq i \leq k}$ a winning strategy for player II , if II always wins the game by choosing $g_i(c_0, \dots, c_i)$ on each round $i \leq k$.

(iii) If a winning strategy for player II exists, then we say II wins $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$ and mark $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$.

(iv) For $a = b = \emptyset$, we define $EF_k((\mathcal{A}, a), (\mathcal{B}, b))$ exactly as above except that $f_0 = \emptyset$. If $a = b = \emptyset$, we notate $EF_k((\mathcal{A}, a), (\mathcal{B}, b)) := EF_k(\mathcal{A}, \mathcal{B})$.

We shall often use the abridgment "EF game" when referring to an Ehrenfeucht-Fraïssé game.

Theorem 2.29. [2, theorem 7.8.]

Suppose L is finite, $k \in \mathbb{N}$, $a \in A^n$ and $b \in B^n$. Then the following are equivalent:

- (i) $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$
- (ii) For all L -formulas $\phi(x)$, $x = (x_1, \dots, x_n)$, of quantifier rank $\leq k$, $\mathcal{A} \models \phi(a)$ if and only if $\mathcal{B} \models \phi(b)$.

Proof. As in [2].

(i) \Rightarrow (ii): By induction on k .

(1) $k = 0$: The definitions of an EF game and the quantifier rank imply directly that for all atomic L -formulas $\phi(x)$, $x = (x_1, \dots, x_n)$, $\mathcal{A} \models \phi(a)$ if and only if $\mathcal{B} \models \phi(b)$. Thus a trivial induction yields that (ii) holds.

(2) $k = p + 1$: By induction on ϕ :

(2.a) ϕ is atomic: By (1).

(2.b) $\phi = \neg\psi$ or $\phi = \psi \wedge \theta$: Directly by the induction assumption.

(2.c) $\phi(x) = \exists x\psi(y, x)$: Now $qr(\phi) = qr(\psi) + 1$ by the definition of the quantifier rank (definition 2.20), so $qr(\psi) = p$. Suppose $\mathcal{A} \models \phi(a)$. Then there is $c \in A$ such that $\mathcal{A} \models \psi(a, c)$. Now suppose player I chooses c on the round 1. Since $II \uparrow EF_p((\mathcal{A}, (a, b)), (\mathcal{B}, (b, d)))$, there is $d \in B$ such that $II \uparrow EF_p((\mathcal{A}, (a, b)), (\mathcal{B}, (b, d)))$. The induction assumption yields then that $\mathcal{A} \models \psi(a, c)$ if and only if $\mathcal{B} \models \psi(b, d)$. Thus we may conclude $\mathcal{B} \models \psi(b, d)$, which implies $\mathcal{B} \models \phi(b)$. On the other hand, symmetry yields that if $\mathcal{B} \models \phi(b)$, then $\mathcal{A} \models \phi(a)$.

(ii) \Rightarrow (i): As follows, we define formula sets Φ_n^k for every $k, n \in \mathbb{N}$, and prove by a simultaneous induction that that each Φ_n^k is finite and consists of formulas with quantifier rank k and no other variables displayed than v_0, \dots, v_n .

$k = 0, n \in \mathbb{N}$: Let $\theta_1, \dots, \theta_n$ list, up to equivalence, all the unquantified L -formulas in which no other variables than v_0, \dots, v_n appear (such a finite list exists by lemma 2.21). Let Φ_n^0 be the finite set of the formulas $\bigwedge_{i \in X} \theta_i \wedge \bigwedge_{i \in Y} \neg \theta_i$ for any $X \subseteq \{1, \dots, n\}$, $Y = \{1, \dots, n\} \setminus X$.

$k = p + 1, n \in \mathbb{N}$: By the induction assumption Φ_{n+1}^p is finite. We enumerate the set Φ_{n+1}^p as $\theta_i(x, v_{n+1})$, $1 \leq i \leq m$. Let Φ_n^k be the set of the formulas

$$\psi_Y(x) = \bigwedge_{i \in Y} \exists v_{n+1} \theta_i(x, v_{n+1}) \wedge \forall v_{n+1} \bigvee_{i \in Y} \theta_i(x, v_{n+1})$$

for any non-empty $Y \subseteq \{1, \dots, m\}$. We note that Φ_n^k is clearly finite and following the application of induction assumption to Φ_{n+1}^p , consists of formulas with quantifier rank k and no other variables displayed than v_0, \dots, v_n .

Now it clearly suffices to prove the following claim.

Claim 1. Suppose $k \in \mathbb{N}$, $a \in A^n$ and $b \in B^n$. Then

- (a) there is $\psi(x) \in \Phi_n^k$ such that $\mathcal{A} \models \psi(a)$
- (b) provided $\psi(x) \in \Phi_n^k$ and $\mathcal{A} \models \psi(a)$ holds iff $\mathcal{B} \models \psi(b)$ holds, then $II \uparrow EF_k((\mathcal{A}, a), (\mathcal{B}, b))$.

We prove claim 1 by induction on k .

$k = 0$: Clear.

$k = p + 1$: Let $\Phi_{n+1}^p = \{\theta_i(x, v_{n+1}) : 1 \leq i \leq m\}$ and define $Y = \{i \in \mathbb{N} : \mathcal{A} \models \exists v_{n+1} \theta_i(a, v_{n+1})\}$. By the induction assumption, $Y \neq \emptyset$ and thus $\psi_Y(x) \in \Phi_n^k$. In addition $\mathcal{A} \models \psi_Y(a)$. This covers (a).

As for (b), there is $\psi(x)_Y \in \Phi_n^k$ such that $\mathcal{A} \models \psi(a)$ by (a). Suppose $\mathcal{A} \models \psi_Y(a)$ holds iff $\mathcal{B} \models \psi_Y(b)$ holds. We describe a winning strategy for II . Let $c_0 \in A \cup B$ be the first move of I . We can assume $c_0 \in A$ as the other case is symmetric. Now $\mathcal{A} \models \psi_Y(a)$ implies that there is $i \in Y$ such that $\mathcal{A} \models \theta_i(a, c_0)$. Since $\mathcal{B} \models \psi_Y(b)$, the structure of ψ_Y yields $\mathcal{B} \models \exists v_{n+1} \theta_i(b, v_{n+1})$, so there is $d \in B$ such that $\mathcal{B} \models \theta_i(b, d)$. Now $\theta_i \in \Phi_{n+1}^p$ and $\mathcal{A} \models \theta_i(a, c_0)$ holds iff $\mathcal{B} \models \theta_i(b, d)$ holds, so the induction assumption yields $II \uparrow EF_k((\mathcal{A}, a, c_0), (\mathcal{B}, b, d))$, which means that II has a winning strategy for the rest of the game. \square Claim 1.

\square

Theorem 2.30. From [8].

Let $n \in \mathbb{N}$ and $L = \{<\}$, where $<$ is a binary relation symbol. Suppose \mathcal{A} and \mathcal{B} are finite linear orders of cardinality $\geq 2^n - 1$. Then $\mathcal{A} \equiv_n \mathcal{B}$.

Proof. The claim is trivial for $n = 0$, so we may assume $n \geq 1$.

Any finite L -model \mathcal{C} that is well-ordered by $<^{\mathcal{C}}$ is isomorphic to some $\mathcal{N}(k) = (\{1, \dots, k\}, <)$, where $(p, q) \in <^{\mathcal{N}(k)}$ if and only if $p < q$ (as in " p is a smaller number than q "). Therefore we may assume $\mathcal{A} = \mathcal{N}(k)$ and $\mathcal{B} = \mathcal{N}(k')$ for some $k, k' \geq 2^n - 1$. In addition, theorem 2.29 implies it is enough to show $II \uparrow EF_n(\mathcal{A}, \mathcal{B})$.

Now suppose $1 \leq m \leq n$ and $(a_1, \dots, a_m) \in A^m$, $(b_1, \dots, b_m) \in B^m$ and $a_1 \leq \dots \leq a_m$. Then we say that $f = \{(a_i, b_i) : 1 \leq i \leq m\}$ has the interval quality if the following holds: $b_1 \leq \dots \leq b_m$ and for any $2 \leq j \leq m-1$, the intervals $[a_i, a_{i+1}]$ and $[b_i, b_{i+1}]$ are mutually either of the same length or both are $\geq 2^{n-m}$, and the same holds for $[0, a_1]$ and $[0, b_1]$ as well as $[a_m, k+1]$ and $[b_m, k'+1]$.

A function $f : X \rightarrow \mathcal{B}$, $X \subseteq \text{dom}(\mathcal{A})$, is a partial isomorphism if and only if f is a bijection and preserves the order $<$, i.e. for all $p, q \in \text{dom}(f)$, $p < q$ iff $f(p) < f(q)$. Therefore, if II is able to follow a strategy $(g_i)_{1 \leq i \leq n}$ such that for any $1 \leq m \leq n$ and $a \in A^m$, $g_m(a)$ has the interval quality, then II clearly wins the game. Hence it suffices to show the following:

Claim 1. There is a strategy $(g_i)_{1 \leq i \leq n}$ for player II such that $g_m(a)$ has the interval quality for any $1 \leq m \leq n$ and $a \in A^m$.

Proof of claim 1: We construct $(g_i)_{1 \leq i \leq n}$ recursively and simultaneously prove claim 1 by induction on $1 \leq m \leq n$.

(1) $m = 1$: By symmetry we may assume I chooses $a_1 \in A$. If $k - a_1 \leq 2^{n-1} - 1$, player II chooses $b_1 = k' - a_1$, and otherwise II chooses $b_1 = a_1$. Function $g_m(a_1)$ clearly has the interval quality.

(2) $m = p + 1$: By symmetry we may assume I chooses $a_{m+1} \in A$, and that it falls in the interval $[c, d]$, where $c = a_i, d = a_{i+1}$ for some $1 \leq i < m$, or $c = 0$ and $d = a_1$, or $d = k + 1$ and $c = a_m$. Let $[c', d']$ be the corresponding interval in B .

Case 1: $d - a_{m+1} < 2^{n-m-1}$. Based on the induction assumption, by choosing $b_{m+1} = d' - (d - a_{m+1})$ player II ensures $[c, a_{m+1}]$ and $[c', b_{m+1}]$, as well as $[a_{m+1}, d]$ and $[b_{m+1}, d']$, have the same lengths.

Case 2: $a_{m+1} - c < 2^{n-m-1}$. Symmetric to case 1.

Case 3: $d - a_{m+1} \geq 2^{n-m-1}$ and $a_{m+1} - c \geq 2^{n-m-1}$. Now $d - c \geq 2^{n-m}$, so the induction assumption yields $d' - c' \geq 2^{n-m}$. Therefore player II is able to choose b_{m+1} so that $d' - b_{m+1} \geq 2^{n-m-1}$ and $b_{m+1} - c' \geq 2^{n-m-1}$.

Hence $g_m(a_1, \dots, a_m)$ has the interval quality. \square Claim 1.

\square

Theorem 2.30 yields the following as an immediate consequence:

Corollary 2.31. Let $L = \{<\}$, where $<$ is a binary relation symbol. Then there is no such L -sentence θ that given a finite, well-ordered L -model \mathcal{M} , $\mathcal{M} \models \theta$ if and only if $|\mathcal{M}|$ is even.

Definition 2.32. From [1], page 10.

Suppose \mathcal{A} is an L -model, $P \in L$ is a binary relation, $R, S \in L$ are unary relation symbols, $L_R := L \setminus \{c \in L : c^{\mathcal{A}} \notin R^{\mathcal{A}}\}$, $L_S := L \setminus \{c \in L : c^{\mathcal{A}} \notin S^{\mathcal{A}}\}$ and \mathcal{R}, \mathcal{S} are L -models, such that $\mathcal{R} \upharpoonright L_R = \mathcal{A} \upharpoonright (R^{\mathcal{A}}, L_R)$ and $\mathcal{S} \upharpoonright L_S = \mathcal{A} \upharpoonright (S^{\mathcal{A}}, L_S)$. A $P^{\mathcal{A}}$ -ranked back-and-forth system in alphabet L between \mathcal{R} and \mathcal{S} is a sequence $(I_j)_{j \in P^{\mathcal{A}}}$ of non-empty sets of partial isomorphisms, such that

(i) if $j \in \text{dom}(P^{\mathcal{A}}) \cup \text{rng}(P^{\mathcal{A}})$ and $f \in I_j$, then $f : U \rightarrow \mathcal{R}$ for some $U \subseteq S^{\mathcal{A}}$, or $f : V \rightarrow \mathcal{S}$ for some $V \subseteq R^{\mathcal{A}}$

(ii) if $P^{\mathcal{A}}(k, j)$, then for every $a \in R^{\mathcal{A}}$ there is $b \in S^{\mathcal{A}}$ such that $f \cup \{(a, b)\} \in I_k$, and for every $b \in S^{\mathcal{A}}$ there is $a \in R^{\mathcal{A}}$ such that $f \cup \{(a, b)\} \in I_k$.

Theorem 2.33. *Suppose \mathcal{A} is a countable L -model and $(I_j)_{j \in P^{\mathcal{A}}}$ is a $P^{\mathcal{A}}$ -ranked back-and-forth system in alphabet L between \mathcal{R} and \mathcal{S} . If $P^{\mathcal{A}}$ has an infinite descending sequence, then \mathcal{R} and \mathcal{S} are isomorphic.*

Proof. We begin by enumerating the sets $\text{dom}(\mathcal{R})$ and $\text{dom}(\mathcal{S})$ as $\text{dom}(\mathcal{R}) = \{a_i : i \in \mathbb{N}\}$ and $\text{dom}(\mathcal{S}) = \{b_i : i \in \mathbb{N}\}$. Let $\{(j_{k+1}, j_k) : k \in \mathbb{N}\} \subseteq P^{\mathcal{A}}$ be an infinite descending sequence. Let us construct a rising chain of partial isomorphisms $f_0 \subseteq f_1 \subseteq \dots$ as follows:

- (i) $f_0 = \emptyset$
- (ii) For an even $k \in \mathbb{N}$, $f_{k+1} = f_k \cup a_i \mapsto b_j$, where $i = \min\{i \in \mathbb{N} : a_i \notin \text{dom}(f_k)\}$, and $j \in \mathbb{N}$, provided by the back-and-forth system, is such that $f_{k+1} \in I_{j_{k+1}}$.
- (iii) For an uneven $k \in \mathbb{N}$, $f_{k+1} = f_i \cup a_i \mapsto b_j$, where $j = \min\{j \in \mathbb{N} : b_j \notin \text{rng}(f_k)\}$, and $i \in \mathbb{N}$, provided by the back-and-forth system, is such that $f_{k+1} \in I_{j_{k+1}}$.

Then $\bigcup_{k \in \mathbb{N}} f_k : \mathcal{R} \rightarrow \mathcal{S}$ is an isomorphism. □

Theorem 2.34. *Suppose L is finite and \mathcal{A} is an L -model. Suppose also that*

- (1) $R, S \in L$, $q \in \mathbb{N}$, $q \geq 1$, $L_R := L \setminus \{c \in L : c^{\mathcal{A}} \notin R^{\mathcal{A}}\}$, $L_S := L \setminus \{c \in L : c^{\mathcal{A}} \notin S^{\mathcal{A}}\}$
- (2) L -models \mathcal{R} and \mathcal{S} are such that $\mathcal{R} \equiv_q \mathcal{S}$, $\mathcal{R} \upharpoonright L_R = \mathcal{A} \upharpoonright (R^{\mathcal{A}}, L_R)$ and $\mathcal{S} \upharpoonright L_S = \mathcal{A} \upharpoonright (S^{\mathcal{A}}, L_S)$
- (3) $P^{\mathcal{A}} = \{(p_0, p_1), \dots, (p_{q-1}, p_q)\} \subseteq A^2$.

Then there is a $P^{\mathcal{A}}$ -ranked back-and-forth system in alphabet L between S and R such that $P^{\mathcal{A}}$ is a linear order in A but $|P^{\mathcal{A}}| = q$.

Proof. First we note that defined as above, $P^{\mathcal{A}}$ is a linear order and $|P^{\mathcal{A}}| = q$.

Since $\mathcal{R} \equiv_q \mathcal{S}$, theorem 2.29 yields that $II \upharpoonright EF_q(\mathcal{R}, \mathcal{S})$. Let $(g_i)_{1 \leq i \leq q}$,

$$g_i : (R^{\mathcal{A}} \cup S^{\mathcal{A}})^i \rightarrow \{f : X \rightarrow \mathcal{S} \mid X \subseteq R^{\mathcal{A}}\}$$

for each $1 \leq i \leq q$, be a winning strategy for player II . We define sequence $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ such that $I_{p_0} = \emptyset$ and for all $1 \leq k \leq q$,

$$I_{p_k} = \{g_k(a_1, \dots, a_k) : (a_1, \dots, a_k) \in (R^{\mathcal{A}} \cup S^{\mathcal{A}})^k\}.$$

We show by induction that for any $0 \leq k \leq q$, $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ is a $P^{\mathcal{A}}$ -ranked back-and-forth system in alphabet L between S and R .

(1) $k = 0$: Clear.

(2) $k = n + 1$: Since the assumptions regarding L , \mathcal{A} , \mathcal{R} and \mathcal{S} match those at the definition 2.32 of back-and-forth systems, it suffices to show that (i) any $f \in I_{p_k}$ is a

partial isomorphism between \mathcal{R} and \mathcal{S} , and (ii) for any $f \in I_{p_n}$, $a \in R^{\mathcal{A}}$, $b \in S^{\mathcal{A}}$ there are $c \in S^{\mathcal{A}}$, $d \in R^{\mathcal{A}}$ such that $f \cup \{(a, c)\}, f \cup \{(b, d)\} \in I_{p_k}$.

(2.i) Given any $(a_1, \dots, a_k) \in (R^{\mathcal{A}} \cup S^{\mathcal{A}})^k$, $g_k(a_1, \dots, a_k)$ is a partial isomorphism between \mathcal{R} and \mathcal{S} by the definition 2.28 (ii) of winning strategies of EF games.

(2.ii) Let $f \in I_{p_n}$. Then by the definition of $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$, there is $(a_1, \dots, a_n) \in (R^{\mathcal{A}} \cup S^{\mathcal{A}})^n$ such that $f = g_n(a_1, \dots, a_n)$. A partial isomorphism must be an injection as otherwise formula $\neg v_0 = v_1$ would create a contradiction with the definition of a partial isomorphism; hence we may assume f^{-1} exists. Now if f is a partial isomorphism $Y \rightarrow \mathcal{R}$, $Y \subseteq S^{\mathcal{A}}$, then f^{-1} is a partial isomorphism $X \rightarrow \mathcal{S}$, $X \subseteq R^{\mathcal{A}}$, so we may assume $f : X \rightarrow \mathcal{S}$, $X \subseteq R^{\mathcal{A}}$. Suppose $a \in R^{\mathcal{A}}$ and $b \in S^{\mathcal{A}}$. Then the definition of $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ yields $h_1 = g_k(a_1, \dots, a_n, a) \in I_{p_k}$ and $h_2 = g_k(a_1, \dots, a_n, b) \in I_{p_k}$, where $c := h_1(a) \in S^{\mathcal{A}}$, $d := h_2(b) \in R^{\mathcal{A}}$, and $h_1 = f \cup \{a, c\}$ as well as $h_2 = f \cup \{b, d\}$. □

Corollary 2.35. *Suppose the assumptions of theorem 2.34 hold otherwise, but the following takes place instead of (2):*

Assume L -models \mathcal{R} and \mathcal{S} are such that $\mathcal{R} \upharpoonright L_R = \mathcal{A} \upharpoonright (R^{\mathcal{A}}, L_R)$ and $\mathcal{S} \upharpoonright L_S = \mathcal{A} \upharpoonright (S^{\mathcal{A}}, L_S)$. Also suppose $f : X \rightarrow S^{\mathcal{A}}$, $X \subseteq R^{\mathcal{A}}$, is a partial isomorphism, such that for all formulas $\phi(x_1, \dots, x_n)$ of quantifier rank $\leq q$ and for all $a \in (\text{dom}(f))^n$, $\mathcal{R} \models \phi(a)$ if and only if $\mathcal{S} \models \phi(f(a))$.

Then there is a $P^{\mathcal{A}}$ -ranked back-and-forth system $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ in alphabet L between \mathcal{S} and \mathcal{R} such that $P^{\mathcal{A}}$ is linear, $|P^{\mathcal{A}}| = q$ and $f \in I_{p_0}$.

Proof. Let $L^* = L \cup \{\underline{a} : a \in \text{dom}(f)\}$. We extend models \mathcal{A} , \mathcal{R} and \mathcal{S} to L^* -models \mathcal{A}^* , \mathcal{R}^* and \mathcal{S}^* by adding interpretations for $\{\underline{a} : a \in \text{dom}(f)\}$ the following way: $\underline{a}^{\mathcal{R}^*} = \underline{a}^{\mathcal{A}^*} = a$ and $\underline{a}^{\mathcal{S}^*} = f(a)$. Now $\mathcal{R}^* \equiv_q \mathcal{S}^*$ so we can apply theorem 2.34 and deduct that there is a $P^{\mathcal{A}^*}$ -ranked back-and-forth system $(I_{p_k}^*)_{p_k \in P^{\mathcal{A}^*}}$ in alphabet L^* between \mathcal{S} and \mathcal{R} such that $P^{\mathcal{A}^*}$ is linear and $|P^{\mathcal{A}^*}| = q$. Without loss of generality, we may assume every $g \in I_{p_k}^*$ is $g : X \rightarrow S^{\mathcal{A}^*}$ for some $X \subseteq R^{\mathcal{A}^*}$. Thus, if $g \in I_{p_k}^*$ for any $0 \leq k \leq q$, then $g \cup f$ is a partial isomorphism by $\mathcal{R}^* \equiv_q \mathcal{S}^*$. Let us define a new sequence $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ so that $I_{p_k} = \{g \cup f : g \in I_{p_k}^*\}$ for all $p_k \in P^{\mathcal{A}}$.

Claim 1. $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ is a $P^{\mathcal{A}}$ -ranked back-and-forth system in alphabet L between \mathcal{S} and \mathcal{R} such that $P^{\mathcal{A}}$ is linear, $|P^{\mathcal{A}}| = q$ and $f \in I_{p_0}$.

Proof of claim 1: By the assumptions of this lemma and the definition of $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$, $P^{\mathcal{A}}$ is linear, $|P^{\mathcal{A}}| = q$ and $f \in I_{p_0}$. Hence, by the definition 2.32 of back-and-forth systems, it suffices to show that (i) any $h \in I_{p_k}$ is a partial isomorphism between \mathcal{R} and \mathcal{S} , and (ii) for any $h \in I_{p_n}$, $a \in R^{\mathcal{A}}$, $b \in S^{\mathcal{A}}$ there are $c \in S^{\mathcal{A}}$, $d \in R^{\mathcal{A}}$ such that $h \cup \{(a, c)\}, h \cup \{(b, d)\} \in I_{p_k}$.

For (i), let $0 \leq k \leq q$. Then every $h \in I_{p_k}$ is a partial isomorphism since it is of the form $g \cup f$ for some $g \in I_{p_k}^*$, and any $g \cup f$, $g \in I_{p_k}^*$, is a partial isomorphism between \mathcal{R} and \mathcal{S} , as was noted above. To show (ii), suppose $P^{\mathcal{A}}(n, k)$, $h \in I_{p_n}$ and $a \in R^{\mathcal{A}}$. Now h is of the form $g \cup f$ for some $g \in I_{p_n}^*$, and $(I_{p_k}^*)_{p_k \in P^{\mathcal{A}^*}}$ being a back-and-forth system,

there is $b \in S^{\mathcal{A}}$ such that $g \cup \{(a, b)\} \in I_{p_k}^*$. The definition of $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ yields then $g \cup f \cup \{(a, b)\} = h \cup \{(a, b) \in I_{p_k}\}$. By symmetry, we may conclude that also for every $b \in S^{\mathcal{A}}$ there is $a \in R^{\mathcal{A}}$ such that $h \cup \{(a, b) \in I_{p_k}\}$. Hence $(I_{p_k})_{p_k \in P^{\mathcal{A}}}$ is a $P^{\mathcal{A}}$ -ranked back-and-forth system in alphabet L between S and R . \square Claim 1. \square

2.3 Classic results of first-order model theory

Theorem 2.36. Compactness Theorem

If T is a first-order theory every finite subset of which has a model, then T has a model.

Proof. See [2]. \square

When referring to the Compactness Theorem, we mean the above formulation. However, there exists also an alternative, equivalent version:

Theorem 2.37. *The Compactness Theorem is equivalent to the following: Let T be a theory and ϕ a sentence such that $T \vdash \phi$. Then there is a finite $U \in T$ such that $U \vdash \phi$.*

Proof. " \Rightarrow " Suppose that if T is a theory every finite subset of which has a model, then T has a model. Let T be a theory and ϕ a sentence such that $T \vdash \phi$. Then there are two options.

(1) T is inconsistent: By assumption, there must now be a finite $U \in T$ such that U is inconsistent. Since U has no model, $U \vdash \psi$ for any sentence ψ , so in particular $U \vdash \phi$.

(2) T is consistent: Now theory $T \cup \{\neg\phi\}$ has no model, so by the assumption, there is a finite $U \subseteq T \cup \{\neg\phi\}$ such that U has no model. Since T is consistent, so is any its subset, so $\neg\phi \in U$. Since U has no model, the sentence

$$\theta := \bigvee \{\neg\psi : \psi \in U \setminus \{\neg\phi\}\} \vee \phi$$

is true in any (L -) model. By notating $U \setminus \{\neg\phi\} = \{\psi_1, \dots, \psi_n\}$ and by using the notation defined under definition 2.12, we observe that given a model \mathcal{A} , $\mathcal{A} \models \theta$ if and only if

$$\mathcal{A} \models \psi_1 \rightarrow (\psi_2 \rightarrow (\dots \psi_{n-1} \rightarrow (\psi_n \rightarrow \phi) \dots)).$$

Thus

$$\vdash \psi_1 \rightarrow (\psi_2 \rightarrow (\dots \psi_{n-1} \rightarrow (\psi_n \rightarrow \phi) \dots)),$$

so $U \setminus \{\neg\phi\} \vdash \phi$, where $U \setminus \{\neg\phi\}$ is finite.

\Leftarrow : Suppose that if T is a theory and ϕ a sentence such that $T \vdash \phi$, then there is a finite $U \in T$ such that $U \vdash \phi$. Let us assume now that T has no model and that ϕ is a sentence. Then $T \vdash \phi \wedge \neg\phi$. By assumption there is a finite $U \in T$ such that $U \vdash \phi \wedge \neg\phi$, so U is inconsistent. Thus we establish that an inconsistent theory has an inconsistent finite subtheory, which to equals to that if T is a theory every finite subset of which is consistent, then T is consistent. Therefore, if T is a theory every finite subset of which has a model, then T has a model. \square

Theorem 2.38. *(Downward) Löwenheim-Skolem Theorem*

If T is a first-order theory that has an infinite model, then it has a countable model.

Proof. See [2]. □

Theorem 2.39. *Robinson Consistency Theorem*

Let T_i be a consistent L_i theory, $i \in \{1, 2\}$. Suppose $T_1 \cap T_2$ is a complete $L_1 \cap L_2$ -theory. Then $T_1 \cup T_2$ is consistent.

Proof. See [6]. □

We exceptionally prove the following theorem in standard, non-relational logic. Since every sentence has an equivalent relational version, so does the interpolant of the below theorem. Further, any sentence equivalent to the interpolant clearly works as an interpolant as well.

We also note that the proof of the theorem uses only definitions expressed prior to the introduction of the restriction to relational logic, which took place after lemma 2.19. The proof doesn't lean on any other prior results than the Compactness Theorem, which is proven in non-relational logic in [2] and therefore can be used below (the relational version of the Compactness Theorem is a trivial consequence of the non-relational one).

Theorem 2.40. *The Craig Interpolation Theorem*

Suppose L_1 and L_2 are countable. Let φ be an L_1 -sentence and σ an L_2 -sentence such that $\varphi \rightarrow \sigma$. Then there is an $L_1 \cap L_2$ -sentence θ (the interpolant) such that $\vdash \varphi \rightarrow \theta$ and $\vdash \theta \rightarrow \sigma$.

Proof. Inspired by [3].

Suppose there is no such θ . Given any alphabets L_1^* and L_2^* and theories T_1 in L_1^* and T_2 in L_2^* , we say that T_1 and T_2 are *separable* if there is a $L_1^* \cap L_2^*$ -sentence ϕ such that $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$. Otherwise, we say that T_1 and T_2 are *inseparable*.

Theories $\{\varphi\}$ and $\{\neg\sigma\}$ are inseparable, since elsehow there was an $L_1 \cap L_2$ -sentence θ such that $\{\varphi\} \vdash \theta$ and $\{\neg\sigma\} \vdash \neg\theta$, in which case we had $\varphi \rightarrow \theta$ and $\theta \rightarrow \sigma$, meaning that θ would work as an interpolant between φ and σ , which we assumed to be impossible.

We introduce a new countably infinite set of constant symbols, C , where $C \cap (L_1 \cup L_2) = \emptyset$. Let $L'_i = L_i \cup C$, $i \in \{1, 2\}$, let $(\phi_i)_{i \in \mathbb{N}}$ be an enumeration of all L'_1 -sentences and $(\psi_i)_{i \in \mathbb{N}}$ of all L'_2 -sentences. Define $T_0 := \{\varphi\}$ and $U_0 := \{\neg\sigma\}$. If now $p = n + 1$ and T_n and U_n are inseparable, then either $T_n \cup \{\phi_n\}$ and U_n are inseparable, or $T_n \cup \{\neg\phi_n\}$ and U_n are inseparable, since otherwise $T_n \cup \{\phi_n\} \vdash \theta$, $U_n \vdash \neg\theta$, $T_n \cup \{\neg\phi_n\} \vdash \xi$ and $U_n \vdash \neg\xi$ for some $L'_1 \cap L'_2$ -sentences θ, ξ , meaning that $T_n \vdash \theta \vee \xi$ and $U_n \vdash \neg\theta \vee \neg\xi$, which contradicts T_n and U_n being separable. We may symmetrically conclude that either T_n and $U_n \cup \{\psi_n\}$ are inseparable, or T_n and $U_n \cup \{\neg\psi_n\}$ are inseparable. Furthermore, if $\phi_n = \exists x \xi(x)$ and $c \in C$ doesn't appear in the sentences of T_n , then for any sentence θ that uses no other vocabulary than what appears on both T_n and U_n , $T_n \cup \{\xi(c)\} \vdash \theta$ implies $T_n \vdash \theta$. Since the equivalent holds for U_n , we can define that

(1) $T_p = T_n \cup \{\theta\} \cup X$, where $\theta = \phi_n$ if $T_n \cup \{\phi_n\}$ is inseparable and $\theta = \neg\phi_n$ otherwise, and $X = \emptyset$, if θ is not of the form $\exists x\xi(x)$, while otherwise, $X = \{\xi(c)\}$ for $c \in C$ that doesn't appear in the sentences of T_n .

(2) $U_p = U_n \cup \{\theta\} \cup Y$, where $\theta = \psi_n$ if $U_n \cup \{\psi_n\}$ is inseparable and $\theta = \neg\psi_n$ otherwise, and $Y = \emptyset$, if θ is not of the form $\exists x\xi(x)$, while otherwise, $Y = \{\xi(c)\}$ for $c \in C$ that doesn't appear in the sentences of U_n .

Now $T_\omega = \bigcup_{i \in \mathbb{N}} T_i$ and $U_\omega = \bigcup_{i \in \mathbb{N}} U_i$ are complete, and also inseparable by the compactness theorem. Also $T_\omega \cap U_\omega$ is complete, since if θ is an $L'_1 \cap L'_2$ -sentence, then either $\theta \in T_\omega$ or $\neg\theta \in T_\omega$ and $\theta \in U_\omega$ or $\neg\theta \in U_\omega$, and due to the inseparability of T_ω and U_ω , either $\theta \in T_\omega$ and $\theta \in U_\omega$, or $\neg\theta \in T_\omega$ and $\neg\theta \in U_\omega$. Furthermore, both T_ω and U_ω must be consistent since an inconsistent theory is separable from any theory.

We build an L'_1 -model \mathcal{M} by using an abbreviated version of the so-called Henkin construction. Define a new relation \sim in C as follows: $c \sim d$ if and only if $T_\omega \vdash c = d$. \sim is clearly an equivalence relation. Then let $[c]$ be the equivalence class of $c \in C$ under \sim . Now T_ω is complete and consistent, so for all relation symbols $R \in L_1$ and function symbols $f \in L_1$, $\#R, \#f = n$, and for all $c_i \sim c'_i$, $1 \leq i \leq n$, $T_\omega \vdash R(c_1, \dots, c_n)$ if and only if $T_\omega \vdash R(c'_1, \dots, c'_n)$ and $T_\omega \vdash f(c_1, \dots, c_n) = f(c'_1, \dots, c'_n)$.

(*) Since $T_\omega \cap U_\omega$ is consistent, we may stricken the definition of \sim as $c \sim d$ if and only if $T_\omega \cup U_\omega \vdash c = d$. Considering that $C \in L'_1 \cap L'_2$, \sim remains identical under this new definition. Furthermore, by the completeness and consistency of U_ω , we see that for all relation symbols $R \in L_1 \cup L_2$ and function symbols $f \in L_1 \cup L_2$, $\#R, \#f = n$, and for all $c_i \sim c'_i$, $1 \leq i \leq n$, $T_\omega \cup U_\omega \vdash R(c_1, \dots, c_n)$ if and only if $T_\omega \cup U_\omega \vdash R(c'_1, \dots, c'_n)$ and $T_\omega \cup U_\omega \vdash f(c_1, \dots, c_n) = f(c'_1, \dots, c'_n)$.

Thus we may consistently define \mathcal{M} as follows:

$$\text{dom}(\mathcal{M}) = \{[c] : c \in C\}$$

For $d \in L'_1 \cup L'_2$, $d^{\mathcal{M}} = [c]$ if $d = c \in T_\omega \cup U_\omega$

For $f \in L'_1 \cup L'_2$, $\#f = n$, $f^{\mathcal{M}}([c_1], \dots, [c_n]) = [f(c_1, \dots, c_n)]$

For $R \in L'_1 \cup L'_2$, $\#R = n$, $([c_1], \dots, [c_n]) \in R^{\mathcal{M}}$ if $R(c_1, \dots, c_n) \in T_\omega \cup U_\omega$

We prove for all L'_1 -sentences ϕ , that $\mathcal{M} \models \phi$ if and only if $\phi \in T_\omega$, by induction on $\phi \in T_\omega$.

ϕ is atomic:

(a) We show first that for all L -terms $t(x_1, \dots, x_n)$, $t([c_1], \dots, [c_n])^{\mathcal{M}} = [t(c_1, \dots, c_n)]$.

This follows directly from the definition for constant symbols, but suppose $t = f(u_1, \dots, u_m)$. Then inductively $t([c_1], \dots, [c_n])^{\mathcal{M}} = f^{\mathcal{M}}(u_1^{\mathcal{M}}([c_1], \dots, [c_n]), \dots, u_m^{\mathcal{M}}([c_1], \dots, [c_n]))$
 $= f^{\mathcal{M}}([u_1(c_1, \dots, c_n)], \dots, [u_m(c_1, \dots, c_n)]) = [f(u_1(c_1, \dots, c_n), \dots, u_m(c_1, \dots, c_n))]$
 $= [t(c_1, \dots, c_n)]$.

(b) $\phi = t = u$: $t = u \in T_\omega$ iff $t \sim u$ iff $[t] = [u]$ iff $t^{\mathcal{M}} = u^{\mathcal{M}}$ iff $\mathcal{M} \models t = u$.

(c) $\phi = R(t_1, \dots, t_n)$: $R(t_1, \dots, t_n) \in T_\omega$ iff $R^{\mathcal{M}}([t_1], \dots, [t_n])$ iff $R^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})$ iff $\mathcal{M} \models R(t_1, \dots, t_n)$.

$\phi = \neg\psi$: Note that $\neg\phi \in T_\omega$ iff $\phi \notin T_\omega$. By the induction assumption, $\phi \notin T_\omega$ iff $\mathcal{M} \not\models \phi$ iff $\mathcal{M} \models \neg\phi$.

$\phi = \psi \wedge \theta$: Directly by the induction assumption.

$\phi = \exists x\psi(x)$: $\phi \in T_\omega$ iff $\psi(c) \in T_\omega$ for some $c \in C$ iff, by the induction assumption, $\mathcal{M} \models \psi(c)$ for some $c \in C$, which takes place iff $\mathcal{M} \models \phi$.

By symmetric measures we can establish that for all L'_2 -sentences ϕ , $\mathcal{M} \models \phi$ if and only if $\phi \in U_\omega$. Hence $\mathcal{M} \models T_\omega \cup U_\omega$, so in particular $\mathcal{M} \models \varphi \wedge \neg\sigma$. This contradicts $\varphi \rightarrow \sigma$. □

In the next theorem we use notation $\varphi(R)$ for a sentence that displays relation symbol R .

Theorem 2.41. *Beth Definability Theorem*

Suppose $R, S \notin L$ are new n -ary relation symbols and $\varphi(R)$ an $L \cup \{R\}$ -sentence. Let

$$\{\varphi(R), \varphi(S)\} \vdash \forall x_1, \dots, \forall x_n (R(x_1, \dots, x_n) \leftrightarrow S(x_1, \dots, x_n)).$$

Then there is an L -formula $\theta(x_1, \dots, x_n)$ such that

$$\{\varphi(R)\} \vdash \forall x_1, \dots, \forall x_n (R(x_1, \dots, x_n) \leftrightarrow \theta(x_1, \dots, x_n)).$$

Proof. The theorem is equivalent to the following formulation:

Claim 1. Suppose $R, S \notin L$ are new n -ary relation symbols, $\varphi(R)$ an $L \cup \{R\}$ -sentence and $c = (c_1, \dots, c_n) \notin L$ new constant symbols. Let

$$\{\varphi(R), \varphi(S)\} \vdash R(c) \leftrightarrow S(c).$$

Then there is an L -formula $\theta(x_1, \dots, x_n)$ such that

$$\{\varphi(R)\} \vdash (R(c) \leftrightarrow \theta(c)).$$

We prove claim 1.

The assumption of claim 1 yields

$$\{\varphi(R), \varphi(S), R(c)\} \vdash S(c),$$

which implies

$$\{\varphi(R), R(c)\} \vdash \varphi(S) \rightarrow S(c),$$

so we establish that

$$\vdash (\varphi(R) \wedge R(c)) \rightarrow (\varphi(S) \rightarrow S(c)).$$

By applying the Craig Interpolation Theorem we see that there is $L \cup \{c\}$ -sentence $\theta(c)$, such that

$$\vdash (\varphi(R) \wedge R(c)) \rightarrow \theta(c) \text{ and } \vdash \theta(c) \rightarrow (\varphi(S) \rightarrow S(c)).$$

Thus

$$\{\varphi(R)\} \vdash R(c) \rightarrow \theta(c) \text{ and } \{\theta(c), \varphi(S)\} \vdash S(c),$$

so

$$\{\varphi(R)\} \vdash R(c) \rightarrow \theta(c) \text{ and } \{\varphi(S)\} \vdash \theta(c) \rightarrow S(c).$$

Because $R, S \notin L$, the right side yields

$$\{\varphi(R)\} \vdash \theta(c) \rightarrow R(c).$$

Therefore

$$\{\varphi(R)\} \vdash R(c) \leftrightarrow \theta(c),$$

so claim 1 and with it the theorem are true. □

Theorem 2.42. *Morley's theorem.*

Suppose L is countable and T is an L -theory such that for some uncountable cardinal κ , T has, up to isomorphism, exactly one model of cardinality κ . Then T has, up to isomorphism, precisely one model of every uncountable cardinality.

Proof. See [7]. □

3 Pseudo-finite model theory

3.1 Introduction

Definition 3.1. Let L be an alphabet in first order logic. We call an L -model \mathcal{M} **pseudo-finite**, in case all the L -sentences true in \mathcal{M} are also true in a finite model.

Many of the fundamental results of first-order model theory can be formulated in terms of the pseudo-finite models, as we will see below. First, however, we look at a few basics of the pseudo-finite model theory.

Definition 3.2. If L be an alphabet in first-order logic, we define

$$\Gamma_L = \{\varphi \in L : \neg\varphi \text{ has no finite model}\}.$$

Lemma 3.3. *From [2].*

Let L be a first-order alphabet and \mathcal{M} an L -model. Then the following statements are equivalent:

- (1) $\mathcal{M} \models \Gamma_L$.
- (2) Every first-order sentence true in \mathcal{M} is true in some finite model.
- (3) There is a set $\{\mathcal{M}_i : i \in I\}$ of finite L -models and an ultrafilter F on I such that

$$\mathcal{M} \equiv \prod_{i \in I} \mathcal{M}_i / F.$$

Proof. (1) \Rightarrow (2): Assume $\mathcal{M} \models \Gamma_L$. If (2) doesn't hold, then there is a sentence φ such that $\mathcal{M} \models \varphi$ but φ is true only in infinite models. In that case $\neg\varphi \in \Gamma_L$, so $\mathcal{M} \models \neg\varphi$, implying that $\mathcal{M} \not\models \varphi$, which is a contradiction.

(2) \Rightarrow (1): Assume every first-order sentence true in \mathcal{M} is true in some finite model. If $\mathcal{M} \not\models \Gamma_L$, then there is $\varphi \in \Gamma_L$ such that $\mathcal{M} \models \neg\varphi$. Now as per the definition of Γ_L , $\neg\varphi$ holds only in infinite models, which is a contradiction.

(3) \Rightarrow (2): Clear, as when (3) holds, every $\varphi \in L$ true in \mathcal{M} has a finite \mathcal{M}_i for some $i \in I$ such that $\mathcal{M}_i \models \varphi$.

(2) \Rightarrow (3): Suppose every first-order sentence true in \mathcal{M} is true in some finite model. Let $T = Thm\{\mathcal{M}\}$ and let I be the set of finite subsets of T . For each $i \in I$, let \mathcal{M}_i be a finite L -model such that $\mathcal{M}_i \models \bigwedge i$. Define

$$X_i = \{j \in I : \mathcal{M}_j \models \bigwedge i\}.$$

Now $\{X_i : i \in I\}$ has the finite intersection property: If $i, j \in I$, then $i \cup j \in I$, and if $k \in I$ is such that $\mathcal{M}_k \models \bigwedge i \cup j$, then $\mathcal{M}_k \models \bigwedge i$ and $\mathcal{M}_k \models \bigwedge j$. Thus

$$X_{i \cup j} \subseteq X_i \cap X_j.$$

Further, $X_i \neq \emptyset$ for all $i \in I$ since $\mathcal{M}_i \models \bigwedge i$, so $\{X_i : i \in I\}$ can be expanded to an ultrafilter $F \subset \mathcal{P}(I)$. Now if $\mathcal{M} \models \varphi$, then

$$\{j \in I : \mathcal{M}_j \models \varphi\} = X_{\{\varphi\}} \in I \subset F,$$

so

$$\prod_{i \in I} \mathcal{M}_i / F \models \varphi.$$

□

By using theory Γ_L we can easily prove "pseudo-finite" versions of many classic model-theoretic results. In several of the cases we only need to use the fact that the pseudo-finite models of an L -theory T are precisely the models of the theory $T \cup \Gamma_L$, and to apply the classic result to theory $T \cup \Gamma_L$.

Definition 3.4. If T is a theory and φ a sentence, then by $T \vdash_{FIN} \varphi$ we mean $T \cup \Gamma_L \vdash \varphi$, where L is precisely the combined alphabet of T and φ .

Having the above definition tied to the precise alphabet L of T and φ is essential. If both $L' \setminus L^*$ and $L^* \setminus L'$ are non-empty, then for all $\varphi \in \Gamma_{L'} \setminus \Gamma_{L^*}$ and $\psi \in \Gamma_{L^*} \setminus \Gamma_{L'}$,

$$\varphi \wedge \psi \in \Gamma_{L' \cup L^*} \text{ but}$$

$\varphi \wedge \psi \notin \Gamma_{L'}$ and

$\varphi \wedge \psi \notin \Gamma_{L^*}$, so

$$\Gamma_{L'} \cup \Gamma_{L^*} \subsetneq \Gamma_{L' \cup L^*}.$$

3.2 Directly adaptable classic results

Compactness Theorem, Löwenheim-Skolem Theorem and Morley's Theorem can be stated directly in terms of pseudo-finite models.

Theorem 3.5. *Finite Compactness Theorem*

If T is a first-order theory every finite subset of which has a pseudo-finite model, then T has a pseudo-finite model.

Proof. We apply the Compactness Theorem to theory $T \cup \Gamma_L$: Suppose $S \subseteq T$ is finite and it has a pseudo-finite model \mathcal{M} . Then $\mathcal{M} \models S \cup S^*$ for any finite $S^* \subseteq \Gamma_L$. Thus, if every finite $S \subseteq T$ has a pseudo-finite model, then every finite $S \subseteq T \cup \Gamma_L$ has a model, which by the Compactness Theorem means that $T \cup \Gamma_L$ has a model \mathcal{N} , and since $\mathcal{N} \models \Gamma_L$, \mathcal{N} is a pseudo-finite model of T . □

When referring to the Finite Compactness Theorem we mean the above version. However, there exists an alternative formulation as in the case of the classic Compactness Theorem.

Theorem 3.6. *Finite Alternative Compactness Theorem*

Let T be a theory and ϕ a sentence such that $T \vdash_{FIN} \phi$. Then there is a finite $U \subseteq T$ such that $U \vdash_{FIN} \phi$.

Proof. Let T be a theory and ϕ a sentence such that $T \cup \Gamma_L \vdash \phi$. By the Compactness Theorem and theorem 2.37, there is a finite $S \subseteq T \cup \Gamma_L$ such that $S \vdash \phi$. Define $U := S \setminus \Gamma_L$, observing that $U \subseteq T$ is also finite. Then $U \cup \Gamma_L \vdash \phi$, meaning $U \vdash_{FIN} \phi$. □

Theorem 3.7. *Finite Löwenheim-Skolem Theorem*

If T is a countable first-order theory and T has a pseudo-finite model, then T has a pseudo-finite countable model.

Proof. Suppose T is a countable first-order theory with a pseudo-finite model \mathcal{M} . Then \mathcal{M} is a model for the theory $T \cup \Gamma_L$, so by the Löwenheim-Skolem Theorem, $T \cup \Gamma_L$ has a countable model \mathcal{N} , and since $\mathcal{N} \models \Gamma_L$, \mathcal{N} is a countable pseudo-finite model of T . □

Theorem 3.8. *Finite Morley's Theorem*

If a countable first-order theory T has, up to isomorphism, exactly one pseudo-finite model in some uncountable cardinality, then it has, up to isomorphism, precisely one pseudo-finite model in every uncountable cardinality.

Proof. Suppose T is a countable L theory with, up to isomorphism, exactly one pseudo-finite model \mathcal{M} in some uncountable cardinality κ . Then L , and consequently Γ_L , consisting of finite subsets of a countable set, must be countable, and \mathcal{M} is the only one model (up to isomorphism) that $T \cup \Gamma_L$ has on κ . Now by Morley's Theorem, $T \cup \Gamma_L$ has precisely one model (up to isomorphism) in every uncountable cardinality, so T has precisely one pseudo-finite model (up to isomorphism) in every uncountable cardinality. \square

3.3 Interpolation

For Craig Interpolation, Beth Definability and Robinson Consistency Theorems we are unable to simply apply the classic theorems to the theory $T \cup \Gamma_L$ and by that to obtain pseudo-finite versions of the theorems. We will see this by counter examples. For the one for Craig's theorem, we need the following lemma, which implies that no sentence of empty vocabulary can characterize any class of finite models such that the cardinalities of the models in the class have no upper boundary.

Lemma 3.9. *Suppose $L = \emptyset$ and θ is an L -sentence with $qr(\theta) \leq k$. Then $\mathcal{M} \models \theta$ if and only if $\mathcal{N} \models \theta$ for any models \mathcal{M}, \mathcal{N} of any alphabet with $|M|, |N| \geq k$.*

Proof. Note that for all models \mathcal{A} , $\mathcal{A} \models \theta$ if and only if $\mathcal{A} \upharpoonright L \models \theta$. Therefore we may assume that \mathcal{M} and \mathcal{N} are models of the empty alphabet. By theorem 2.29 it suffices to prove that player II has a winning strategy in any $EF_k(\mathcal{M}, \mathcal{N})$.

Now $f : X \rightarrow \mathcal{N}$, $X \subseteq M$, is a partial isomorphism if it is an injection, so II can clearly win by a strategy $(g_i)_{1 \leq i \leq k}$ that satisfies the following:

$i = 1$: If I chooses $c_1 \in M$ then II chooses any $c_1 \mapsto d_1$, $d_1 \in N$, and if I chooses $c_1 \in N$ then II chooses any $d_1 \mapsto c_1$, $d_1 \in M$.

$i = n + 1 \leq k$: Suppose f_n is the partial isomorphism $X \rightarrow \mathcal{N}$, $X \subseteq M$, developed by the previous rounds. Let $c_{n+1} \in M \cup N$ be the choice of I on round $n + 1$. If $c_{n+1} \in \text{dom}(f_n) \cup \text{rng}(f_n)$ then II chooses $f_{n+1} = f_n$. Suppose $c_{n+1} \notin \text{dom}(f_n) \cup \text{rng}(f_n)$. If $c_{n+1} \in M$ then II chooses any $f_n \cup \{c_{n+1}, d_{n+1}\}$ where $d_{n+1} \in N \setminus \text{rng}(f_n)$ which exists since $|f_n| \leq n$ but $|N| \geq k$. If $c_{n+1} \in N$ then II chooses any $f_n \cup \{d_{n+1}, c_{n+1}\}$ where $d_{n+1} \in M \setminus \text{dom}(f_n)$ which exists since $|f_n| \leq n$ but $|M| \geq k$. \square

It follows from corollary 2.33, that there is no sentence of the empty alphabet that would characterize even cardinality in finite models (i.e. there is no such θ that for a finite model, $\mathcal{M} \models \theta$ if and only if $|M|$ is even). However, our main application of the lemma

is to prove that the Craig Interpolation Theorem cannot be directly stated in terms of pseudo-finite models by replacing the relation \vdash by \vdash_{FIN} .

Example 3.10. *There are finite alphabets L_1 and L_2 and L_i sentences ϕ_i , $i \in \{1, 2\}$ such that $\vdash_{FIN} \phi_1 \rightarrow \phi_2$ but no $L_1 \cap L_2$ -sentence θ satisfies both $\vdash_{FIN} \phi_1 \rightarrow \theta$ and $\vdash_{FIN} \theta \rightarrow \phi_2$.*

Proof. Let $L_1 = \{R\}$ and $L_2 = \{S\}$ where R and S are distinct binary relation symbols. Let ϕ_1 say that R is an equivalence relation with all classes of size two and let ϕ_2 say $\neg (S \text{ is an equivalence relation with all classes of size two except for one of size one})$. Then $\vdash_{FIN} \phi_1 \rightarrow \phi_2$.

Contrary to the claim of the example, suppose now that there is an $L_1 \cap L_2$ -sentence θ such that $\vdash_{FIN} \phi_1 \rightarrow \theta$ and $\vdash_{FIN} \theta \rightarrow \phi_2$. Then θ is a sentence of the empty alphabet. Let $qr(\theta) \leq 2k$ and let \mathcal{M} be an L_1 -model of cardinality $2k$ such that $\mathcal{M} \models \phi_1$. In addition, let \mathcal{N} be an L_2 -model of cardinality $2k + 1$ such that $\mathcal{N} \models \neg \phi_2$. Then $\mathcal{M} \models \theta$ but $\mathcal{N} \models \neg \theta$. However, lemma 3.10 yields $\mathcal{M} \models \theta$ if and only if $\mathcal{N} \models \theta$, a contradiction. \square

Example 2.11 shows that the classic version of Craig Interpolation Theorem does not hold if restricted to pseudo-finite models. However, by using $\Gamma_{L_1} \cup \Gamma_{L_2}$ instead of $\Gamma_{L_1 \cup L_2}$ we obtain an alternative formulation:

Theorem 3.11. *Finite Craig Interpolation Theorem*

Suppose φ is an L_1 -sentence and ψ an L_2 -sentence such that

$$\Gamma_{L_1} \cup \Gamma_{L_2} \vdash \varphi \rightarrow \psi.$$

Then there is an $L_1 \cap L_2$ -sentence θ such that $\vdash_{FIN} \varphi \rightarrow \theta$ and $\vdash_{FIN} \theta \rightarrow \psi$.

Proof. By the Alternative Compactness Theorem (theorem 2.37), there is a finite $U = (S_1 \cup S_2) \subseteq \Gamma_{L_1} \cup \Gamma_{L_2}$ such that

$$S_1 \cup S_2 \vdash \varphi \rightarrow \psi$$

and $S_1 \subseteq \Gamma_{L_1}$ and $S_2 \subseteq \Gamma_{L_2}$. Now

$$S_1 \cup S_2 \cup \{\varphi\} \vdash \psi,$$

so

$$S_1 \cup \{\varphi\} \vdash \bigwedge S_2 \rightarrow \psi,$$

so

$$\vdash (\bigwedge S_1 \wedge \varphi) \rightarrow (\bigwedge S_2 \rightarrow \psi).$$

By the Craig Interpolation Theorem there is an $L_1 \cap L_2$ -sentence θ such that $\vdash (\bigwedge S_1 \wedge \varphi) \rightarrow \theta$ and $\vdash \theta \rightarrow (\bigwedge S_2 \rightarrow \psi)$. Then

$$S_1 \vdash \varphi \rightarrow \theta \quad \text{and} \quad \{\theta\} \vdash \bigwedge S_2 \rightarrow \psi,$$

so

$$S_1 \vdash \varphi \rightarrow \theta \quad \text{and} \quad S_2 \cup \{\theta\} \vdash \psi,$$

so

$$S_1 \vdash \varphi \rightarrow \theta \quad \text{and} \quad S_2 \vdash \theta \rightarrow \psi,$$

so

$$\Gamma_{L_1} \vdash \varphi \rightarrow \theta \quad \text{and} \quad \Gamma_{L_2} \vdash \theta \rightarrow \psi.$$

Therefore $\vdash_{FIN} \varphi \rightarrow \theta$ and $\vdash_{FIN} \theta \rightarrow \psi$.

□

Alike Craig's theorem, the classic version of Robinson's cannot be stated directly in terms of pseudo-finite models. We see this more easily by re-formulating the theorem:

Theorem 3.12. *Alternative Robinson Consistency Theorem*

Let T_i be a L_i theory, $i \in \{1, 2\}$, such that each T_i has a model and all the $L_1 \cap L_2$ -models of $T_1 \cap T_2$ are elementary equivalent. Then $T_1 \cup T_2$ has a model.

Proof. Directly by the Robinson Consistency Theorem, since $T_1 \cap T_2$ is complete if and only if all the $L_1 \cap L_2$ -models of $T_1 \cap T_2$ are elementary equivalent.

□

Example 3.13. *There are alphabets L_i and L_i -theories T_i , $i \in \{1, 2\}$, such that each T_i has a pseudo-finite model and all the pseudo-finite $L_1 \cap L_2$ -models of $T_1 \cap T_2$ are elementary equivalent, but $T_1 \cup T_2$ has no pseudo-finite model.*

Proof. Let L_i and ϕ_i , $i \in \{1, 2\}$, be as in example 2.11. We define theories T_1 and T_2 such that both include the sentences

$$\exists x_0, \dots, \exists x_n \bigwedge_{0 \leq i < j \leq n} x_i \neq x_j$$

for all $n \in \mathbb{N}$, in addition to which $\phi_1 \in T_1$ and $\neg \phi_2 \in T_2$. Now T_1 is an L_1 -theory and is satisfied by $\mathcal{M} = (\mathbb{N}, R)$, where $R^{\mathcal{M}}$ is an equivalence relation with all classes of size two. Alike, T_2 is an L_2 -theory and is satisfied by $\mathcal{M} = (\mathbb{N}, S)$, where $S^{\mathcal{M}}$ is an equivalence relation with all classes of size two except one of size one. The models \mathcal{M} and \mathcal{N} are shown pseudo-finite by example 3.8, which is proven independently of this example. Now all the $L_1 \cap L_2 = \emptyset$ -models of $T_1 \cap T_2$ are elementary equivalent by lemma 3.10, but there is no pseudo-finite model that would satisfy $T_1 \cup T_2$.

□

We obtain, however, the following:

Theorem 3.14. *Finite Robinson Consistency Theorem*

Let T_i be L_i -theories, $i \in \{1, 2\}$, with pseudo-finite models. Suppose that the pseudo-finite $L_1 \cap L_2$ -models of $T_1 \cap T_2$ are elementary equivalent. Then there is an $L_1 \cup L_2$ -model \mathcal{M} such that $\mathcal{M} \models T_1 \cup T_2$, and $\mathcal{M} \upharpoonright L_1$ and $\mathcal{M} \upharpoonright L_2$ are pseudo-finite.

Proof. Theories $T_1 \cup \Gamma_{L_1}$ and $T_2 \cup \Gamma_{L_2}$ satisfy now the conditions of the classic Robinson Consistency Theorem. Thus we obtain a model \mathcal{M} such that $\mathcal{M} \models T_1 \cup T_2 \cup \Gamma_{L_1} \cup \Gamma_{L_2}$. Now $\mathcal{M} \models T_1 \cup \Gamma_{L_1}$ and $\mathcal{M} \models T_2 \cup \Gamma_{L_2}$, so $\mathcal{M} \models T_1 \cup T_2$ and $\mathcal{M} \upharpoonright (L_i)$, $i \in \{1, 2\}$, is pseudo-finite. \square

Also the classic Beth Definability Theorem fails when restricted to pseudo-finite models, as we see below:

Example 3.15. *There are an alphabet L , an n -ary relation symbol R and an $L \cup \{R\}$ -sentence $\varphi(R)$ such that for a new n -ary relation symbol $S \notin L$,*

$$\{\varphi(R), \varphi(S)\} \vdash_{FIN} \forall x_1, \dots, \forall x_n (R(x_1, \dots, x_n) \leftrightarrow S(x_1, \dots, x_n)),$$

but there is no L -formula $\theta(x_1, \dots, x_n)$ such that

$$\{\varphi(R)\} \vdash_{FIN} \forall x_1, \dots, \forall x_n (R(x_1, \dots, x_n) \leftrightarrow \theta(x_1, \dots, x_n)).$$

Proof. Let us state the claim in an equivalent way:

There are an alphabet L , an n -ary relation symbol R and an $L \cup \{R\}$ -sentence $\varphi(R)$ such that for a new n -ary relation symbol and a new constant symbol $S, c \notin L$,

$$\Gamma_{L \cup \{R, S, c\}} \cup \{\varphi(R), \varphi(S)\} \vdash R(c) \leftrightarrow R'(c),$$

but there is no L -formula $\theta(x_1, \dots, x_n)$ such that

$$\Gamma_{L \cup \{R, RSc\}} \cup \{\varphi(R)\} \vdash (R(c) \leftrightarrow \theta(c)).$$

Let $L = \{<\}$ where $<$ is a binary relation symbol and let L -sentence ψ say that the relation $<$ is a strict total order. In addition, suppose $R, S, c \notin L$, where R, S are unary relation symbols and $\varphi(R)$ the following $L \cup \{R\}$ -sentence:

$$\begin{aligned} & (\forall v_0 (\forall v_1 (v_0 \neq v_1 \rightarrow v_0 < v_1) \rightarrow \neg R(v_0)) \\ & \wedge \forall v_0 \forall v_1 (((v_0 < v_1) \wedge (\nexists v_2 (v_0 < v_2 < v_1))) \rightarrow ((\neg R(v_0) \leftrightarrow R(v_1))). \end{aligned}$$

Let \mathcal{M} be now a finite $L \cup \{R\}$ -model such that $\mathcal{M} \models \psi \wedge \varphi(R)$. Then $<^{\mathcal{M}}$ is a well-order and for all $a \in M$, $R(a)$ if and only if a has an even rank in $<^{\mathcal{M}}$. Define $\phi(R) := \psi \wedge \varphi(R)$. Then $\phi(R) \wedge \phi(S) \vdash_{FIN} R(c) \leftrightarrow S(c)$, but if $\theta(x)$ is an L -formula such that $\varphi(R) \vdash_{FIN} \forall x (\theta(x) \leftrightarrow R(x))$, then $\theta(a)$ says that the element a has an even rank in $<^{\mathcal{M}}$. Now if an L -formula $\sigma(x)$ defines x as the element with the highest rank in $<$ then $\forall x (\sigma(x) \rightarrow \theta(x))$ defines the cardinality of a finite, ordered $\{<\}$ -model as even, which by theorem 2.30 is impossible. \square

Similar Finite Craig Interpolation, we obtain a pseudo-finite version for the Beth Definability Theorem by splitting $\Gamma_{L \cup \{R, R'\}}$ into $\Gamma_{L \cup \{R\}} \cup \Gamma_{L \cup \{R'\}}$.

Theorem 3.16. *Finite Beth Definability Theorem.*

Suppose $R, S \notin L$ are new n -ary relation symbols and $\varphi(R)$ an $L \cup \{R\}$ -sentence. Let

$$\Gamma_{L \cup \{R\}} \cup \Gamma_{L \cup \{S\}} \cup \{\varphi(R), \varphi(S)\} \vdash \forall x_1, \dots, \forall x_n (R(x_1, \dots, x_n) \leftrightarrow S(x_1, \dots, x_n)).$$

Then there is an L -formula $\theta(x_1, \dots, x_n)$ such that

$$\{\varphi(R)\} \vdash_{FIN} \forall x_1, \dots, \forall x_n (R(x_1, \dots, x_n) \leftrightarrow \theta(x_1, \dots, x_n)).$$

Proof. The theorem is equivalent to the following formulation:

Claim 1. Suppose $R, S \notin L$ are new n -ary relation symbols, $\varphi(R)$ an $L \cup \{R\}$ -sentence and $c = (c_1, \dots, c_n) \notin L$ new constant symbols. Let

$$\Gamma_{L \cup \{R, c\}} \cup \Gamma_{L \cup \{S, c\}} \cup \{\varphi(R), \varphi(S)\} \vdash R(c) \leftrightarrow S(c).$$

Then there is an L -formula $\theta(x_1, \dots, x_n)$ such that

$$\{\varphi(R)\} \vdash_{FIN} (R(c) \leftrightarrow \theta(c)).$$

We prove claim 1.

By the assumption,

$$\Gamma_{L \cup \{R, c\}} \cup \Gamma_{L \cup \{S, c\}} \vdash (\varphi(R) \wedge R(c)) \rightarrow (\varphi(S) \rightarrow S(c)).$$

By applying the Finite Craig Interpolation Theorem here we obtain an $L \cup \{c\}$ -sentence $\theta(c)$ such that

$$\vdash_{FIN} (\varphi(R) \wedge R(c)) \rightarrow \theta(c) \text{ and } \vdash_{FIN} \theta(c) \rightarrow (\varphi(S) \rightarrow S(c)),$$

so

$$\Gamma_{L \cup \{R, c\}} \vdash (\varphi(R) \wedge R(c)) \rightarrow \theta(c) \text{ and } \Gamma_{L \cup \{S, c\}} \vdash \theta(c) \rightarrow (\varphi(S) \rightarrow S(c)).$$

This yields

$$\Gamma_{L \cup \{R, c\}} \cup \{\varphi(R)\} \vdash R(c) \rightarrow \theta(c) \text{ and } \Gamma_{L \cup \{S, c\}} \cup \{\varphi(S)\} \vdash \theta(c) \rightarrow S(c).$$

Since $S, R \notin L$, the right side of the above implies

$$\Gamma_{L \cup \{R, c\}} \cup \{\varphi(R)\} \vdash \theta(c) \rightarrow R(c).$$

Therefore

$$\Gamma_{L \cup \{R, c\}} \cup \{\varphi(R)\} \vdash R(c) \leftrightarrow \theta(c),$$

so

$$\{\varphi(R)\} \vdash_{FIN} R(c) \leftrightarrow \theta(c).$$

□ Claim1.

□

3.4 A Lindström Theorem

First-order logic has a characterization, the Lindström Theorem, stating that any extension of first-order logic that satisfies both the Compactness Theorem and the Löwenheim-Skolem Theorem has no sentence that wouldn't be equivalent to some first-order sentence [1]. As we shall see, under certain conditions the same holds when we restrict to pseudo-finite models.

Definition 3.17. Inspired by [4] and [5]

(i) An **abstract logic** is a triple $L = (S, F, \models)$, where $\models \subseteq S \times F$. Elements of the class S are called the models of L , elements of F are called the sentences of L , and the relation \models is called the satisfaction relation of L . If $\mathcal{M} \in S$ and $\varphi \in F$, then $\mathcal{M} \models \varphi$ means that φ is true in \mathcal{M} and that \mathcal{M} is a model of φ , while $\mathcal{M} \not\models \varphi$ means the opposite.

(ii) L is **closed under conjunction** if for all $\varphi, \psi \in F$ there is $\theta \in F$ such that for all $\mathcal{M} \in S$, $\mathcal{M} \models \theta$ if and only if $\mathcal{M} \models \varphi$ and $\mathcal{M} \models \psi$.

(iii) Suppose every $\mathcal{M} \in S$ is a τ -model for some alphabet τ , in the sense of definition 2.7. Then L is **closed under negation** if for all $\varphi \in F$ and for all τ there is $\theta \in F$ such that for all τ -models, $\mathcal{M} \in S$, $\mathcal{M} \models \theta$ if and only if $\mathcal{M} \not\models \varphi$.

(iv) $L_{\omega\omega} = (Str, FO, \models)$ denotes the normal (relational) first-order logic, where Str consists of all models and FO of all formulas of any alphabet. If $L^* = (S^*, F^*, \models^*)$ and $\mathcal{M}, \mathcal{N} \in S^*$, we say that \mathcal{M} and \mathcal{N} are isomorphic if they are isomorphic in $L_{\omega\omega}$. In addition, we say that L^* is **closed under isomorphisms**, if whenever $\varphi \in F$, \mathcal{M} and \mathcal{N} being isomorphic in $L_{\omega\omega}$ implies that $\mathcal{M} \models \varphi$ if and only if $\mathcal{N} \models \varphi$.

(v) For $L = (S, F, \models)$, any $T \subseteq F$ is called a **theory**.

Definition 3.18. (i) If $L^* = (S, F, \models^*)$, we say that L^* **relativizes** if for all $\varphi \in F$ and all unary relation symbols $U \in \tau$ there is $\varphi^{(U)} \in F$ such that for all $\mathcal{M} \in S$, such that $\mathcal{M} \upharpoonright U^{\mathcal{M}}$ is a model, $\mathcal{M} \models^* \varphi^{(U)}$ if and only if $\mathcal{M} \upharpoonright U^{\mathcal{M}} \models^* \varphi$.

(ii) If $L = (S, F, \models)$ and $L^* = (S, F^*, \models^*)$ are abstract logics, $L \leq L^*$ **on finite models** means that for all $\varphi \in F$ there is $\psi \in F^*$ such that for all finite $\mathcal{M} \in S$, $\mathcal{M} \models \varphi$ if and only if $\mathcal{M} \models \psi$. In this case we also say that every $\varphi \in L$ is definable in L^* .

In this case we also say that every $\varphi \in F^*$ is definable in L' on finite models.

(iii) If $L^* = (S, F, \models^*)$ and $\mathcal{M} \in S$, then let us call \mathcal{M} **L^* -pseudo-finite**, if every $\varphi \in F$ true in \mathcal{M} is true also in a finite $\mathcal{N} \in S$. If the referenced logic L^* is clear by association, then we simply say that \mathcal{M} is pseudo-finite.

As with first-order logic, any set of sentences is called a theory. Given an alphabet τ , an abstract logic $L^* = (S, F, \models^*)$ satisfies the **Finite Compactness Theorem** if whenever T is a theory in L^* , every finite subset of which has a pseudo-finite model, then T has a pseudo-finite model. L^* satisfies the **Finite Löwenheim-Skolem Theorem**, if whenever T is a countable theory in L^* and T has a pseudo-finite model, then T has a countable pseudo-finite model.

Definition 3.19. Let $L^* = (S, F, \models^*)$ be an abstract logic, T a theory in L^* and $\varphi \in F$. Then by $T \vdash_{FIN} \varphi$ we mean that for every pseudo-finite $\mathcal{M} \in S$, if $\mathcal{M} \models \psi$ for all $\psi \in T$, then $\mathcal{M} \models \varphi$. If $T = \emptyset$, then we write $\vdash_{FIN} \varphi$.

Theorem 3.20. *Theorem 3 of [2].*

Let $L_{\omega\omega} = (Str, FO, \models)$ and let $L^* = (Str, F, \models^*)$ be an abstract logic such that for all $\theta \in F$ there is a finite alphabet τ such that $\mathcal{M} \models \theta$ or $\mathcal{M} \models \neg\theta$ for all τ -models \mathcal{M} . Suppose $FO \subseteq F$, $L_{\omega\omega} \leq L^*$ on finite models, L^* is closed under negation, conjunction and isomorphisms, it relativizes and $\models \subseteq \models^*$. Suppose L^* satisfies the Finite Compactness Theorem and the Finite Löwenheim-Skolem Theorem. Then $L^* \leq L_{\omega\omega}$ on finite models.

Proof. Suppose $\theta \in F$ is not first-order definable on finite models and τ is a finite alphabet such that $\mathcal{M} \models \theta$ or $\mathcal{M} \models \neg\theta$ for all τ -models \mathcal{M} . Let $\phi_i^{m,k}$, $i \leq q(m, k)$, form a list of τ -formulas of quantifier rank $\leq k$ such that

- (i) if $i \leq q(m, k)$, then no other variables than v_0, \dots, v_n appear free in ϕ_i
- (ii) for any τ -formula ψ , $qr(\psi) \leq k$, in which no other variables than v_0, \dots, v_n appear free, there is $i \leq q(m, k)$ such that ψ is equivalent with $\phi_i^{m,k}$.

Such a list $\phi_i^{m,k}$, $i \leq q(m, k)$, exists by lemma 2.21, since τ is finite by assumption. Now let

$$\phi^k = \bigwedge \{ \phi_i^{0,k} : i \leq q(0, k), \vdash_{FIN} \theta \rightarrow \phi_i^{0,k} \}.$$

The set defining sentence ϕ^k is non-empty as the sentences $\phi_n^{0,k}$, $n < q(0, k)$, include some that are true in all τ -models, such as the sentence $\exists x(x = x)$, or someone equivalent to it. Therefore the sentence ϕ^k exists. Now $\vdash_{FIN} \theta \rightarrow \phi^k$, since ϕ^k is of the form $\bigwedge \Psi$, where $\vdash_{FIN} \theta \rightarrow \varphi$ for all $\varphi \in \Psi$. On the other hand $\not\vdash_{FIN} \phi^k \rightarrow \theta$, as otherwise we had $\vdash_{FIN} \theta \leftrightarrow \phi^k$, meaning that ϕ^k defines θ , which would go against the assumption that θ is not first-order definable on finite models. Thus we can choose a finite $\mathcal{A}_k \models \phi^k \wedge \neg\theta$. Let

$$\psi^k = \bigwedge \{ \phi_n^{0,k} : n < q(0, k), \mathcal{A}_k \models \phi_n^{0,k} \}.$$

If it were now that $\vdash_{FIN} \theta \rightarrow \neg\psi^k$, then $\neg\psi^k \in \Psi$, where $\phi^k = \bigwedge \Psi$, implying that $\vdash_{FIN} \phi^k \rightarrow \neg\psi^k$, which is in contradiction with $\mathcal{A}_k \models \phi^k \wedge \psi^k$. Therefore we can choose a finite $\mathcal{B}_k \models \theta \wedge \psi^k$. Now the $L_{\omega\omega}$ -sentences true in \mathcal{B}_k with $\leq k$ variable symbols are precisely the same ones that are true in \mathcal{A}_k , by the definition of ψ^k , so $\mathcal{A}_k \equiv_k \mathcal{B}_k$ (where \equiv_k refers to $L_{\omega\omega}$ -sentences only). However, $\mathcal{A}_k \models \neg\theta$ and $\mathcal{B}_k \models \theta$.

Given sets X, Y , and $m \in \mathbb{N}$, let ${}^Y X := \{s : X \rightarrow Y\}$, and for $q, p \in \mathbb{N}$, $q, p \geq 1$, let ${}^q Y := \{s : \{1, \dots, q\} \rightarrow Y\}$, ${}^X q := \{s : X \rightarrow \{1, \dots, q\}\}$ and ${}^q p := \{s : \{1, \dots, q\} \rightarrow \{1, \dots, p\}\}$.

Let $R, S \notin \tau$ be new unary relation symbols. Based on the assumption made for language L^* , relativizations $\varphi^{(S)}$ and $(\neg\varphi)^{(R)}$ are included in F for any $\varphi \in F$. For every τ -atomic formula $\psi(x_1, \dots, x_k)$, let $\Phi_{v_i \rightarrow v_{i+1}}(\psi)$ be the formula

$$\psi^{(R)}(v_1, v_3, \dots, v_{2k-1}) \leftrightarrow \psi^{(S)}(v_2, v_4, \dots, v_{2k}),$$

and let $\Theta_{i \rightarrow i+1}(\psi)(x_1, \dots, x_{2k})$ be

$$\bigwedge \left\{ \Phi_{v_i \rightarrow v_{i+1}}(\psi) \left((x_1/v_{s(1)})(x_2/v_{s(1)+1}) \dots (x_{2k-1}/v_{s(2k)-1})(x_{2k}/v_{s(2k)}) \right) : s \in {}^k k \right\}.$$

Then let H denote the set of all τ -atomic formulas and let $\Upsilon_n(v_1 \dots, v_{2n})$ be the conjunction of the set

$$\left\{ \Theta_{i \rightarrow i+1}(\psi)(v_1 \dots, v_{2n}) : \psi \in H \right\}.$$

Now then, if \mathcal{M} is a τ -model such that $\mathcal{M} \upharpoonright R^{\mathcal{M}} \subseteq \mathcal{M}$, $\mathcal{M} \upharpoonright S^{\mathcal{M}} \subseteq \mathcal{M}$ and $\mathcal{M} \models \Upsilon_n(a_1, b_1, \dots, a_n, b_n)$ for $a_1, \dots, a_n \in R^{\mathcal{M}}$, $b_1, \dots, b_n \in S^{\mathcal{M}}$, then $a_i \mapsto b_i$ is a partial isomorphism $X \rightarrow \mathcal{M} \upharpoonright S^{\mathcal{M}}$, $X \subseteq \mathcal{M} \upharpoonright R^{\mathcal{M}}$.

Let P and F be relation symbols such that $\#P = 2$ and $\#F = 5$. Let T be the $\tau \cup \{R, S, P, F\}$ -theory consisting of the following sentences:

1. For all $n \in \mathbb{N}$ and all n -ary τ -function symbols f , $\forall x_1 \dots \forall x_n ((R(x_1, \dots, x_n) \rightarrow R(f(x_1, \dots, x_n)) \wedge (S(x_1, \dots, x_n) \rightarrow S(f(x_1, \dots, x_n))))$
2. $\theta^{(S)}$
3. $(\neg\theta)^{(R)}$
4. $\forall x \forall y (((P(x, y) \wedge P(y, x)) \rightarrow x = y) \wedge ((P(x, y) \wedge P(y, z)) \rightarrow P(x, z)) \wedge (P(x, y) \vee P(y, x)))$
5. $\varphi_n \wedge \psi_n$, $n \in \mathbb{N} \setminus \{0, 1\}$, where

$$\begin{aligned} \psi_n &:= \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_{2n} \forall z_1 \dots \forall z_{2n} \left(\left(\bigwedge_{1 \leq i \leq n-1} P(x_i, x_{i+1}) \right. \right. \\ &\quad \left. \left. \wedge \bigwedge_{2 \leq i \leq 2n} F(y_{i-1}, z_{i-1}; y_i, z_i; x_i) \right) \rightarrow \bigwedge_{1 \leq i \leq 2n} (R(y_i) \wedge S(z_i) \wedge \Upsilon_{2n}(y_1, z_1, \dots, y_{2n}, z_{2n})) \right) \end{aligned}$$

and

$$\varphi_n := \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_{2n} \forall z_1 \dots \forall z_{2n-1} \exists z_{2n} \left(\left(\bigwedge_{1 \leq i \leq n-1} P(x_i, x_{i+1}) \right. \right.$$

$$\begin{aligned}
& \bigwedge_{2 \leq i \leq 2n-1} F(y_{i-1}, z_{i-1}; y_i, z_i; x_i) \rightarrow F(y_{2n-1}, z_{2n-1}; y_{2n}, z_{2n}; x_{2n}) \\
& \wedge \forall x_1 \dots \forall x_n \forall z_1 \dots \forall z_{2n} \forall y_1 \dots \forall y_{2n-1} \exists y_{2n} \left(\left(\bigwedge_{1 \leq i \leq n-1} P(x_i, x_{i+1}) \right. \right. \\
& \left. \left. \bigwedge_{2 \leq i \leq 2n-1} F(y_{i-1}, z_{i-1}; y_i, z_i; x_i) \right) \rightarrow F(y_{2n-1}, z_{2n-1}; y_{2n}, z_{2n}; x_{2n}) \right)
\end{aligned}$$

6. φ for all $\varphi \in \Gamma_{L^*}$.

Suppose now a $\tau \cup \{R, S, P, F\}$ -model \mathcal{M} satisfies $U \subseteq T$, where U includes all the sentences clauses 1-4 and a finitely many of the sentences in clauses 5 and 6. Define $L_R := L \setminus \{c \in L : c^{\mathcal{M}} \notin R^{\mathcal{M}}\}$ and $L_S := L \setminus \{c \in L : c^{\mathcal{M}} \notin S^{\mathcal{M}}\}$. Now clause 1 implies that \mathcal{R}, \mathcal{S} are τ -models, when we define them so that $\mathcal{R} \upharpoonright L_R = \mathcal{M} \upharpoonright (R^{\mathcal{M}}, L_R)$ and $\mathcal{S} \upharpoonright L_S = \mathcal{M} \upharpoonright (S^{\mathcal{M}}, L_S)$. Furthermore, for every set X , $|X| = |\text{dom}(\mathcal{R})|$, \mathcal{R} is isomorphic to some model \mathcal{R}^* where $\text{dom}(\mathcal{R}^*) = X$, we may assume $\{c^{\mathcal{S}} : c \in L\} \subseteq \text{dom}(\mathcal{R})$. In addition, by automorphism, we may assume $c^{\mathcal{R}} = c^{\mathcal{S}}$ for all $c \in L$.

Clause 4 ensures that $P^{\mathcal{M}}$ is a linear order, and clause 5 that there is a $P^{\mathcal{M}}$ -ranked back-and-forth system in alphabet τ between \mathcal{R} and \mathcal{S} .

Now for every $k \in \mathbb{N}$, let L^{**} -model \mathcal{C}_k be such that $\text{dom}(\mathcal{C}_k) = \text{dom}(\mathcal{A}_k) \cup \text{dom}(\mathcal{B}_k)$, $\mathcal{C}_k \upharpoonright \text{dom}(\mathcal{A}_k, L) = \mathcal{A}_k$, $\mathcal{C}_k \upharpoonright \text{dom}(\mathcal{B}_k, L) = \mathcal{B}_k$, $R^{\mathcal{C}_k} = \text{dom}(\mathcal{A}_k)$ and $S^{\mathcal{C}_k} = \text{dom}(\mathcal{B}_k)$. Based on theorem 2.34, we can interpret P in \mathcal{C}_k so that for every $n \in \mathbb{N}$ there is such a $k \in \mathbb{N}$ that there is a $P^{\mathcal{C}_k}$ -ranked back-and-forth system $(I_j)_{j \in P^{\mathcal{C}_k}}$ in τ between \mathcal{A}_k and \mathcal{B}_k , such that

- (i) $P^{\mathcal{C}_k} = \{(p_0, p_1), \dots, (p_{n-1}, p_n)\}$ is a linear order
- (ii) $I_{j_0} = \emptyset$.

(*) As for the relation symbol F , we interpret it so that $F^{\mathcal{C}_k}(a, b, c, d, p)$ if and only if $p = p_j$ for some $j \in \{1, \dots, n\}$, and there is $f \in I_{p_{j-1}}$ such that $a \in \text{dom}(f)$, $f(a) = b$ and $f \cup (c, d) \in I_{p_j}$.

Now \mathcal{C}_k satisfies sentences 1, 2 and 3 of T since $\text{dom}(\mathcal{C}_k) = \text{dom}(\mathcal{A}_k) \cup \text{dom}(\mathcal{B}_k)$, $\mathcal{C}_k \upharpoonright \text{dom}(\mathcal{A}_k, L) = \mathcal{A}_k$, $\mathcal{C}_k \upharpoonright \text{dom}(\mathcal{B}_k, L) = \mathcal{B}_k$, $R^{\mathcal{C}_k} = \text{dom}(\mathcal{A}_k)$ and $S^{\mathcal{C}_k} = \text{dom}(\mathcal{B}_k)$, $\mathcal{A}_k \models \neg\theta$ and $\mathcal{B}_k \models \theta$. Sentence 4 holds since $P^{\mathcal{C}_k}$ is a linear order and any number of the sentences in 6. hold as \mathcal{C}_k is finite. In addition, for every $n \in \mathbb{N} \setminus \{0, 1\}$, there is a $k \in \mathbb{N}$ such that $\mathcal{C}_k \models \varphi_m \wedge \psi_m$ for every $\varphi_m \wedge \psi_m$ of 5. with $m \leq n$, since $\varphi_m \wedge \psi_m$, $m \leq n$, are true in \mathcal{C}_k precisely if F is interpret as in (*) and \mathcal{C}_k includes a $P^{\mathcal{C}_k}$ -ranked back-and-forth system $(I_j)_{j \in P^{\mathcal{C}_k}}$ in τ between \mathcal{A}_k and \mathcal{B}_k , such that $(I_j)_{j \in P^{\mathcal{C}_k}}$ satisfies the conditions (i) and (ii), which \mathcal{C}_k will do for a $k \in \mathbb{N}$ big enough, as we saw above.

Thus any finite $U \subset T$ can be expanded into a finite $T^* \subset T$ such that, for some $k \in \mathbb{N}$, $\mathcal{C}_k \models T^*$, which implies $\mathcal{C}_k \models U$. Therefore the Finite Compactness Theorem yields that there is a model \mathcal{C} for the whole T . By the Finite Löwenheim-Skolem theorem, we may

take that \mathcal{C} is countable. Let $\mathcal{C} \upharpoonright (R^{\mathcal{C}}, L) =: \mathcal{R}$ and $\mathcal{C} \upharpoonright (S^{\mathcal{C}}, L) =: \mathcal{S}$. Now \mathcal{R} and \mathcal{S} are isomorphic by theorem 2.33. However, $\mathcal{R} \models \neg\theta$ and $\mathcal{S} \models \theta$, which is a contradiction. \square

4 Characterizing pseudo-finiteness

4.1 Existence of a finite model

In this section we prove theorems stating exact conditions on whether a sentence has a finite model. We also illustrate a construction of a finite model for a sentence that has one but displays no function symbols, using the substructure of a given model satisfying the sentence.

Definition 4.1. Let ψ be a formula.

- (i) We define the set of **subformulas** of ψ , $Sub(\psi)$, as follows:
 $\psi \in Sub(\psi)$
 If $\phi \in Sub(\psi)$ and $\theta \in Sub(\phi)$ then $\theta \in Sub(\psi)$
 If $\phi \in Sub(\psi)$ and $\phi \in \{\neg\theta, \forall x\theta, \exists x\theta\}$, then $\theta \in Sub(\psi)$
 If $\phi \in Sub(\psi)$ and $\phi = \theta \wedge \sigma$ then $\theta, \sigma \in Sub(\psi)$.
 (ii) If $\phi \in Sub(\psi)$ and $\theta \in Sub(\phi)$ then we notate $\phi \in Sub(\psi, \theta)$
 If $\phi \in Sub(\psi)$, we notate $\phi \leq \psi$, and if $\phi \leq \psi$ but $\phi \neq \psi$, we notate $\phi < \psi$.
 (iii) The set of **existential subformulas** of ψ , $ESub(\psi)$, is the set of those subformulas of ψ that are of the form $\exists x\phi$ with x free in ϕ .
 If $\phi \in Sub(\psi)$ and $\theta \in Sub(\phi)$ then $ESub(\psi, \theta) := Sub(\psi, \theta) \cap ESub(\psi)$.

Definition 4.2. Let ψ be a formula.

- (i) We say ψ is in the **quantifiers-first-form** if every subformula of ψ that has bound variables is of the form $\exists x\phi$ or $\forall x\phi$.
- (ii) If ψ is in the quantifiers-first-form, then $\phi \in Sub(\psi)$ is **obtained by no other than quantifier removal** if ϕ is in the quantifiers-first-form.
- (iii) If ψ is in the quantifiers-first-form, then $\phi \in Sub(\psi)$ is **the last subformula of ψ obtained by no other than quantifier removal**, if ϕ is unquantified but any $\theta \in Sub(\psi, \phi)$, $\theta \neq \phi$, is quantified.

Lemma 4.3. *For every formula ψ there is an equivalent formula that is in the quantifiers-first-form.*

Proof. We first observe that for all L -formulas $\theta(x_i)$ and ϕ , where the variable symbol x_i does not appear free in ϕ , the following holds:

$$\vdash \neg\forall x_i\theta \leftrightarrow \exists x_i\neg\theta$$

$$\vdash \neg\exists x_i\theta \leftrightarrow \forall x_i\neg\theta$$

$$\vdash \phi \wedge \forall x_i \theta \leftrightarrow \forall x_i (\phi \wedge \theta) \quad \text{and}$$

$$\vdash \phi \wedge \exists x_i \theta \leftrightarrow \exists x_i (\phi \wedge \theta).$$

Proof of the lemma by induction on ψ .

(1) ψ is atomic: ψ has no subformulas with bound variables so ψ is itself in the quantifiers-first-form.

(2) $\psi = \forall x \phi$ or $\psi = \exists x \phi$: Directly by the induction assumption.

(3) $\psi = \neg \phi$: We may now assume ϕ is in the quantifiers-first-form.

If ϕ is unquantified then so is ψ , and thus by definition, ψ is in the quantifiers-first-form.

If $\phi = \forall x \theta$ then θ is in the quantifiers-first-form, $qr(\theta) = n$, and $\vdash \psi \leftrightarrow \exists x \neg \theta$, so by induction assumption, $\vdash \psi \leftrightarrow \exists x \sigma$ for some σ that is in the quantifiers-first-form; thus ψ is equivalent to a formula in the quantifiers-first-form.

If $\phi = \exists x \theta$ then θ is in the quantifiers-first-form, $qr(\theta) = n$, and $\vdash \psi \leftrightarrow \forall x \neg \theta$, so by induction assumption, $\vdash \psi \leftrightarrow \forall x \sigma$ for some σ that is in the quantifiers-first-form; thus ψ is equivalent to a formula in the quantifiers-first-form.

(4) $\psi = \phi \wedge \theta$: We may now assume ϕ and θ are in the quantifiers-first-form. Now all the variable symbols that occur bounded in ϕ and θ belong to a set $\{v_i : 0 \leq i \leq p\}$ for some $p \geq 1$. Suppose ϕ^* is a formula obtained from ϕ by replacing each variable symbol v_i that occurs bounded in ϕ by variable v_{i+2p} (i.e. every appearance of v_i , even in quantifiers $\exists v_i$ or $\forall v_i$, is replaced by v_{i+2p}), and θ^* is a formula obtained from θ by replacing each variable symbol v_i that occurs bounded in θ by variable v_{i+4p} . Then clearly $\vdash \phi \leftrightarrow \phi^*$ and $\vdash \theta \leftrightarrow \theta^*$. Now $\phi^* = w\sigma$ and $\theta^* = w'\tau$ for some unquantified formulas σ and τ and sign sequences w and w' , each consisting of signs of the forms $\exists x$ and $\forall y$ where $x, y \in \{v_i : 2p \leq i \leq 3p\}$ for the variables appearing in w and $x, y \in \{v_i : 4p \leq i \leq 5p\}$ for the variables appearing in w' . Clearly $\vdash \phi^* \wedge \theta^* \leftrightarrow ww'(\sigma \wedge \tau)$, where $ww'(\sigma \wedge \tau)$ is a formula in the quantifiers-first-form. In addition, $\vdash \phi \leftrightarrow \phi^*$ and $\vdash \theta \leftrightarrow \theta^*$ so $\vdash \psi \leftrightarrow ww'(\sigma \wedge \tau)$. \square

Definition 4.4. Let $B \subseteq X^n$.

(i) For $1 \leq i \leq n$, the i -th **projection** of B , $Pr_i B$, is the set

$$\{b_i : (b_1, \dots, b_n) \in B\}.$$

(ii) For $1 \leq i \leq n$, the **early part** of B until i , $B_{\leq i}$, is the set

$$\{(b_1, \dots, b_i) : (b_1, \dots, b_n) \in B\}.$$

(iii) The **collapse** of B , $Col(B)$ is the set

$$\left\{ b_i : 1 \leq i \leq n, (b_1, \dots, b_n) \in B \right\}.$$

Definition 4.5. Suppose \mathcal{M} is an L -model.

(i) Let $\exists y\psi(x, y)$, $x = (x_1, \dots, x_n)$ be an L -formula. **The set of Skolem functions** of ψ in \mathcal{M} , $Skolem(\psi, \mathcal{M})$, consists of functions $f : \{a \in M^n : \mathcal{M} \models \exists\psi(a, y)\} \rightarrow M$ that satisfy the following: If $a \in dom(f)$, then $\mathcal{M} \models \psi(a, f(a))$. If $n = 0$ then $Skolem(\psi, \mathcal{M}) = \{a \in M : \mathcal{M} \models \psi(a)\}$

(ii) Let φ be an L -sentence in quantifiers-first-form, $\psi(x)$, $x = (x_1, \dots, x_{n-1})$, a subformula of φ obtained by no other than quantifier removal and let $\{\theta_1, \dots, \theta_n\} \subseteq Sub(\varphi, \psi)$ be maximal such that θ_2 has one more free variable than θ_1 , and if $n \geq 2$ and $1 \leq i < j \leq n$, then $\theta_i < \theta_j$ and θ_j has one more free variable than θ_i . Further, list $ESub(\varphi, \psi) = \{\phi_1, \dots, \phi_k\}$, where $\phi_i < \phi_j$ when $1 \leq i < j \leq k$, and for each $1 \leq i \leq k$, let $f_{\phi_i} \in Skolem(\psi, \mathcal{M})$. If $A \subseteq M^n$, then a **Skolem closure** of A from φ to ψ by $f_{\phi_1}, \dots, f_{\phi_k}$, notated as $(\bar{A}, \varphi, \psi, f_{\phi_1}, \dots, f_{\phi_k})$, is the set

$$\bigcup_{i \in \mathbb{N}} A_i,$$

where $A_0 \subseteq A_1 \subseteq \dots \subseteq M^n$ satisfies the following:

$A_0 = A$.

For $i \geq 1$, A_i is the smallest set that holds the following:

If $1 \leq j \leq n$ and $\theta_j \notin ESub(\varphi, \psi)$, then

$$(a_1, \dots, a_j, b, a_{j+2}, \dots, a_n) \in A_i$$

for every $(a_1, \dots, a_n) \in A_{i-1}$ and $b \in Col(A_{i-1}) \cup Col((A_i)_{\leq j-1})$.

If $1 \leq j \leq n$ and $\theta_j \in ESub(\varphi, \psi)$, then

$$(a_1, \dots, a_j, f_{\theta_j}(a_1, \dots, a_j)) \in (A_i)_{\leq j}$$

for every $(a_1, \dots, a_{j-1}) \in (A_i)_{\leq j-1}$.

Definition 4.6. Let \mathcal{M} be an L -model and φ an L -sentence in quantifiers-first-form. Given a subformula $\psi(x)$, $x = (x_0, \dots, x_n)$, of φ obtained by no other than quantifier removal, $REC(\psi, \mathcal{M}, \varphi) \subseteq \mathcal{P}(M^n)$ is the set of **realizing element collections** of ψ in (\mathcal{M}, φ) , if the following holds:

(i) If ψ is a sentence, then $REC(\psi, \mathcal{M}, \varphi) = \emptyset$

(ii) If $\forall y\psi = \phi$ or $\exists y\psi = \phi$, where ϕ is also a subformula of φ and y is bounded in ψ , then $REC(\psi, \mathcal{M}, \varphi) = REC(\phi, \mathcal{M}, \varphi)$

(iii) If $\forall y\psi = \phi$, where ϕ is also a subformula of φ and y is free in ψ , and γ is the set of L -constant symbols in φ , then

$$REC(\psi, \mathcal{M}, \varphi) = \{A \times (Col(A) \cup \{c^\mathcal{M} : c \in \gamma\}) : A \in REC(\phi, \mathcal{M}, \varphi)\},$$

except if $REC(\phi, \mathcal{M}, \varphi) = \emptyset$, in which case

$$REC(\psi, \mathcal{M}, \varphi) = \{\{a, c^\mathcal{M} : c \in \gamma\} : a \in M\}$$

(iv) If $\exists y\psi = \phi$, where ϕ is also a subformula of φ and y is free in $\psi =: \phi_k$, and if $ESub(\varphi, \psi) = \{\phi_1, \dots, \phi_k\}$, then $REC(\psi, \mathcal{M}, \varphi)$ is the set

$$\left\{ (\overline{A \times f_{\phi_k} A}, \varphi, \psi, f_{\phi_1}, \dots, f_{\phi_k}) : A \in REC(\phi, \mathcal{M}, \varphi), f_{\phi_i} \in Skolem(\phi_i, \mathcal{M}), \phi_i \in ESub(\varphi, \phi) \right\}.$$

Lemma 4.7. *Suppose \mathcal{M} is an L -model, φ an L -sentence in quantifiers-first-form, $\mathcal{M} \models \varphi$ and τ is the set of L -symbols appearing in φ . Let θ be the last subformula of φ obtained by no other than quantifier removal, let $B \in REC(\theta, \mathcal{M}, \varphi)$ and let A be the collapse of B . Suppose $\mathcal{A} := \mathcal{M} \upharpoonright (A, \tau) \subseteq \mathcal{M} \upharpoonright \tau$. Then for all $\phi(x) \in Sub(\varphi)$, $\phi \geq \theta$, $x = (x_1, \dots, x_n)$, and $a \in B_{\leq n}$, $\mathcal{A} \models \phi(a)$ if $\mathcal{M} \models \phi(a)$.*

Proof. By induction on $\phi \in Sub(\varphi)$, $\phi \geq \theta$.

(1) $\phi = \theta$: By $\mathcal{A} \subseteq \mathcal{M}$ since θ is unquantified.

(2) $\phi = \forall y\sigma$ or $\phi = \exists y\sigma$ and y is bounded in σ : Directly by the induction assumption.

(3) $\phi = \forall y\sigma$ and y is free in σ : Suppose $\mathcal{M} \models \phi(a)$. Then $\mathcal{M} \models \sigma(a, b)$ for any $b \in M$, so $\mathcal{M} \models \sigma(a, b)$ for any $b \in A$. Hence the induction assumption yields $\mathcal{A} \models \sigma(a, b)$ for any $b \in A$, which implies $\mathcal{A} \models \phi(a)$.

(4) $\phi = \exists y\sigma$ where y is free in σ : Suppose $\mathcal{M} \models \phi(a)$. Then $\mathcal{M} \models \sigma(a, b)$ for some $b \in M$. Now σ has $n + 1$ free variables, so $fB_{\leq n} \subseteq Pr_{n+1}B$ for some $f \in Skolem(\phi, \mathcal{M})$, so $\mathcal{M} \models \sigma(a, f(a))$ where $(a, f(a)) \in B_{\leq n+1}$. Therefore the induction assumption yields $\mathcal{A} \models \sigma(a, f(a))$ and subsequently $\mathcal{A} \models \phi(a)$. □

Theorem 4.8. *Suppose L has no function symbols and L -sentence φ is in quantifiers-first-form. Then φ has a finite model if and only if it has a model \mathcal{M} such that there is a finite $A \in REC(\psi, \mathcal{M}, \varphi)$, where ψ is the last subformula of φ obtained by no other than quantifier removal.*

Proof. " \Leftarrow ": Suppose φ has a model \mathcal{M} such that there is a finite $B \in REC(\psi, \mathcal{M}, \varphi)$, where ψ is the last subformula of φ obtained by no other than quantifier removal. Let τ be the set of L -symbols appearing in φ and let A be the collapse of B . Then A is non-empty, so τ having no function symbols, $\mathcal{M} \upharpoonright (A, \tau) \subseteq \mathcal{M} \upharpoonright \tau$, and subsequently $\mathcal{A} = \mathcal{M} \upharpoonright (A, \tau)$ is a model. Now lemma 4.7 yields that for all $\phi(x) \in Sub(\varphi)$, $\phi \geq \theta$, $x = (x_1, \dots, x_n)$, and $a \in B$, $\mathcal{A} \models \phi(a)$ if $\mathcal{M} \models \phi(a)$. Hence $\mathcal{M} \models \varphi$ implies $\mathcal{A} \models \varphi$, where \mathcal{A} is finite.

" \Rightarrow ": We make a counter claim: For any L -model \mathcal{M} satisfying φ and for ψ the last subformula of φ obtained by no other than quantifier removal, every $A \in REC(\psi, \mathcal{M}, \varphi)$ is infinite. Then the collapse of A is also infinite, meaning that every L -model \mathcal{M} that satisfies φ has an infinite subset and is consequently infinite. This contradicts the assumption that φ has a finite model. □

Theorem 4.9. *Suppose L has no function symbols and L -sentence φ is in quantifiers-first-form. Then φ has a finite model if and only if it has a model \mathcal{M} such that the following holds:*

Let τ is the set of L -symbols appearing in φ and let θ be the last subformula of φ obtained by no other than quantifier removal. Then there is a finite $B \in REC(\theta, \mathcal{M}, \varphi)$, such that if A is the collapse of B , then $\mathcal{A} := \mathcal{M} \upharpoonright (A, \tau) \subseteq \mathcal{M} \upharpoonright \tau$, \mathcal{A} is finite and $\mathcal{A} \models \varphi$.

Proof. If the conditions hold then φ clearly has a finite model. To show the other direction, suppose φ has a finite model. Let τ is the set of L -symbols appearing in φ . Let θ be the last subformula of φ obtained by no other than quantifier removal, let $B \in REC(\theta, \mathcal{M}, \varphi)$ be finite as provided by theorem 4.8, and let A be the collapse of B . Since L has no function symbols, $\mathcal{M} \upharpoonright (A, \tau) \subseteq \mathcal{M} \upharpoonright \tau$. Define $\mathcal{A} := \mathcal{M} \upharpoonright (A, \tau)$. Then \mathcal{A} is finite. In addition, lemma 4.7 yields that for all $\phi(x) \in Sub(\varphi)$, $\phi \geq \theta$, $x = (x_1, \dots, x_n)$, and $a \in B_{\leq n}$, $\mathcal{A} \models \phi(a)$ if $\mathcal{M} \models \phi(a)$. Therefore $\mathcal{M} \models \varphi$ implies $\mathcal{A} \models \varphi$. \square

Lemma 4.10. Suppose \mathcal{M} is an L -model, φ an L -sentence in quantifiers-first-form and $\mathcal{M} \models \varphi$. Let ψ a subformula of φ obtained by no other than quantifier removal and let $A \in REC(\psi, \mathcal{M}, \varphi)$. Then $\mathcal{M} \models \psi(a)$ for all $a \in A$.

Proof. By induction on ψ , a subformula of φ obtained by no other than quantifier removal. The induction goes "downward": Based on the assumption that the claim holds for ϕ , a subformula of φ obtained by no other than quantifier removal such that $\psi < \phi$, we prove it holds for ψ .

- (1) ψ is a sentence: By the assumption $\mathcal{M} \models \varphi$, since in this case ψ is equivalent to φ .
- (2) $\forall y \psi = \phi$ or $\exists y \psi = \phi$, where ϕ is also a subformula of φ and y is bounded in ψ : Then $\vdash \psi \leftrightarrow \phi$ and $REC(\psi, \mathcal{M}, \varphi) = REC(\phi, \mathcal{M}, \varphi)$, so the induction claim returns to the induction assumption.
- (3) $\forall y \psi = \phi$, where ϕ is also a subformula of φ and y is free in ψ : Suppose γ is the set of L -constant symbols in φ . If $REC(\phi, \mathcal{M}, \varphi) = \emptyset$, then

$$REC(\psi, \mathcal{M}, \varphi) = \{\{b, c^{\mathcal{M}} : c \in \gamma\} : b \in M\}.$$

Now $\vdash \phi \leftrightarrow \varphi$ and $\mathcal{M} \models \varphi$ so $\mathcal{M} \models \forall y \psi(y)$, which yields $\mathcal{M} \models \psi(a)$ for any $a \in \{b, c^{\mathcal{M}} : c \in \gamma\}$ for any $b \in M$.

If $REC(\phi, \mathcal{M}, \varphi) \neq \emptyset$ then

$$REC(\psi, \mathcal{M}, \varphi) = \{A \times (Col(A) \cup \{c^{\mathcal{M}} : c \in \gamma\}) : A \in REC(\phi, \mathcal{M}, \varphi)\}.$$

Let $A \in REC(\phi, \mathcal{M}, \varphi)$. Now the induction assumption yields $\mathcal{M} \models \phi(a)$ for all $a \in A$. Therefore $\mathcal{M} \models \psi(a, b)$ for any $a \in A, b \in M$, so $\mathcal{M} \models \psi(a)$ for any

$$a \in A \times (Col(A) \cup \{c^{\mathcal{M}} : c \in \gamma\}).$$

- (4) $\exists y \psi = \phi$, where ϕ is also a subformula of φ and y is free in ψ =: ϕ_k : Let $ESub(\varphi, \psi) = \{\phi_1, \dots, \phi_k\}$. Then $REC(\psi, \mathcal{M}, \varphi)$ is the set

$$\left\{ (\overline{A \times f_{\phi_k} A}, \varphi, \psi, f_{\phi_1}, \dots, f_{\phi_k}) : A \in REC(\phi, \mathcal{M}, \varphi), f_{\phi_i} \in Skolem(\phi_i, \mathcal{M}), \phi_i \in ESub(\varphi, \phi) \right\}.$$

Let $A \in REC(\phi, \mathcal{M}, \varphi)$. The induction assumption yields $\mathcal{M} \models \phi(a)$ for any $a \in A$. Thus the definition of a Skolem closure implies $\mathcal{M} \models \psi(a)$ for any

$$a \in (\overline{A \times f_{\phi_k} A}, \varphi, \psi, f_{\phi_1}, \dots, f_{\phi_k})$$

and any $f_{\phi_i} \in Skolem(\phi_i, \mathcal{M})$ and $\phi_i \in ESub(\varphi, \phi)$. □

From here on we use the following notation: if $\psi(x)$, $x = (x_1, \dots, x_n)$, is a formula, \mathcal{M} an L -model and $a \in M^n$, then by $\psi(a)$ we mean $\mathcal{M} \models \psi(a)$ if the model $\psi(a)$ refers to is clear from context.

Lemma 4.11. *A consistent L -sentence φ is true in infinite models only, if and only if there is an L -formula $\sigma(x_1, \dots, x_n, y_1, \dots, y_k)$ such that for all L -models \mathcal{M} satisfying φ there is $(a_i, b_i)_{i \in \mathbb{N}}$, $a_i \in M^n$, such that for all $i \in \mathbb{N}$,*

- (i) $\sigma(a_i, b_i)$
- (ii) for any $c \in \{c \in M^k : \sigma(a_{i+1}, c)\}$,

$$Col(\{c\}) \not\subseteq Col(\{a_0, \dots, a_i\}).$$

Proof. " \Leftarrow ": If the conditions hold then for any model \mathcal{M} satisfying φ , clearly $a_i \neq a_j$ if $i \neq j$, so M has an infinite subset and is therefore infinite itself.

" \Rightarrow ":

Claim 1. It suffices to prove the " \Rightarrow " of the lemma for alphabets L that have no function symbols.

Proof of claim 1. Let $\psi(x)$, $x = (x_1, \dots, x_k)$, be an L -formula and f an n -ary function symbol that appears in ψ and let $R \notin L$ be a new $n+1$ -ary relation symbol. Suppose we obtain ψ' from φ by replacing every occurrence of $f(z) = y$, $(z, y) = (z_1, \dots, z_n, y)$ any variable symbols, by $R(z, y)$, and suppose $\psi^* = \psi' \wedge \sigma$, where σ is the formula

$$\forall x_1 \dots \forall x_n \exists y R(x_1, \dots, x_n, y)$$

$$\wedge \forall x_1 \dots \forall x_n \forall y \forall z' (R(x_1, \dots, x_n, y) \wedge (R(x_1, \dots, x_n, z') \rightarrow z' = y)).$$

Let \mathcal{M} be an L -model and \mathcal{M}^* an $(L \setminus \{f\}) \cup \{R\}$ -model such that $\mathcal{M}^* \upharpoonright L \setminus \{f\} = \mathcal{M} \upharpoonright L \setminus \{f\}$ and for all $(a_1, \dots, a_n, b) \in M^{n+1}$, $R^{\mathcal{M}^*}(a_1, \dots, a_n, b)$ iff $f^{\mathcal{M}}(a_1, \dots, a_n) = b$. Then clearly for all $a \in M^k$, $\mathcal{M} \models \psi(a)$ if and only if $\mathcal{M}^* \models \psi^*(a)$. Using the same method for an arbitrary L -formula $\phi(x)$, $x = (x_1, \dots, x_n)$, we can obtain an $(L \setminus \{f\}) \cup \{R\}$ -formula ϕ^* which satisfies $\mathcal{M} \models \phi(a)$ iff $\mathcal{M}^* \models \phi^*(a)$ for all $(a_1, \dots, a_n, b) \in M^n$.

Applying the same reasoning for a potentially multiple times we reach an alphabet τ that is obtained from L by replacing every L -function symbol f by a $\#f$ -ary relation symbol R_f and a τ -model \mathcal{M}' , $\mathcal{M}' \upharpoonright L \cap \tau = \mathcal{M} \upharpoonright L \cap \tau$ such that

- (a) for all n -ary function symbols f and all $(a_1, \dots, a_n, b) \in M^{n+1}$, $R_f^{\mathcal{M}'}(a_1, \dots, a_n, b)$ iff $f^{\mathcal{M}}(a_1, \dots, a_n) = b$

(b) for an arbitrary L -formula $\phi(x)$, $x = (x_1, \dots, x_n)$, we can obtain an τ -formula ϕ' which satisfies $\mathcal{M} \models \phi(a)$ iff $\mathcal{M}' \models \phi(a)'$ for all $(a_1, \dots, a_n, b) \in M^n$.

Suppose now that given any formula ψ and model \mathcal{M} , ψ' and \mathcal{M}' mean the τ -formula and the τ -model that are obtained from ψ and \mathcal{M} by the above technique. Further, within this paragraph, suppose that if we refer to a formula ψ' or a model \mathcal{M}' , then they are obtained from some ψ and \mathcal{M} by the aforesaid technique. Note that the operators $\phi \mapsto \phi'$ and $\mathcal{M} \mapsto \mathcal{M}'$ are injective, so given ψ' and \mathcal{M}' , the formula ϕ and model \mathcal{M} they attribute to are unambiguous. Therefore we may state the following: there is an L -formula $\psi(x_1, \dots, x_n, y_1, \dots, y_k)$ such that for any L -model \mathcal{M} satisfying φ there is $(a_i)_{i \in \mathbb{N}}$, $a_i \in M^n$, satisfying conditions (1) and (2), if there is a τ -formula $\psi'(x_1, \dots, x_n, y_1, \dots, y_k)$ such that for any τ -model \mathcal{M}' satisfying φ' there is $(a_i)_{i \in \mathbb{N}}$, $a_i \in (M')^n$, that satisfies (1) and (2). Hence the claim " \Rightarrow " of the lemma is equivalent to the claim that is otherwise identical but restricts to alphabets that have no function symbols. \square Claim 1.

Hence we may assume from here on that L has no function symbols. We assume φ is true in infinite models only and that \mathcal{M} is an L -model that satisfies φ . Let ϕ be the last subformula of φ obtained by no other than quantifier removal. Then by theorem 4.8, every $A \in REC(\phi, \mathcal{M}, \varphi)$ is infinite. Since $REC(\phi, \mathcal{M}, \varphi) \subseteq REC(\psi, \mathcal{M}, \varphi)$ for all $\psi \in Sub(\varphi, \phi)$ and $REC(\varphi, \mathcal{M}, \varphi) = \emptyset$, there is a $\psi \in Sub(\varphi, \phi)$ such that $REC(\psi, \mathcal{M}, \varphi)$ consists of infinite sets only but for any $\sigma \in Sub(\varphi, \psi)$, $\sigma \neq \psi$, $REC(\sigma, \mathcal{M}, \varphi)$ has a finite element. Suppose now ψ is such.

Let ξ be a subformula of φ . If $\forall y \psi = \xi$, or if $\exists y \psi = \xi$ and y is bounded in ψ , then $REC(\psi, \mathcal{M}, \varphi) = REC(\xi, \mathcal{M}, \varphi)$, which goes against the definition of ψ . However, as a sentence φ must be quantified, so we can conclude $\exists y \psi = \psi^*$ for some $\psi^* \in Sub(\varphi)$, where y is free in ψ .

Now φ cannot be equivalent to a sentence of the form $\exists x_0, \dots, \exists x_n \psi$ since if it was then every $A \in REC(\psi, \mathcal{M}, \varphi)$ would be finite. Instead φ equals to $w\psi$ for some sign sequence w consisting of signs of the form $\forall x$ and $\exists x$ that bind variables in ψ . Let θ be the formula obtained from $w\psi$ by removing all signs of the form $\forall x$ from w . Then let $\{p_1, \dots, p_n\} \subset \{1, \dots, n+k\}$ and $\{m_1, \dots, m_k\} = \{1, \dots, n+k\} \setminus \{p_1, \dots, p_n\}$ be such that $\theta = \exists x_{m_1} \dots \exists x_{m_k} \psi(x_{p_1}, \dots, p_n)$. (*)

Let now $ESub(\varphi, \psi) = (\phi_1, \dots, \phi_k)$, let $f_{\phi_i} \in Skolem(\phi_i, \mathcal{M})$ for each $i \in \{1, \dots, k\}$ and let $B_0 \in REC(\exists x_n \psi, \mathcal{M}, \varphi)$ be finite. Further, let and $A_0 := B_0 \times f_{\phi_k} B_0$. Then A_0 is finite. Now $(\overline{A_0}, \varphi, \psi, f_{\phi_1}, \dots, f_{\phi_k})$ is infinite and $(\overline{A_0}, \varphi, \psi, f_{\phi_1}, \dots, f_{\phi_k}) = \bigcup_{i \in \mathbb{N}} A_i$ for a chain $A_0 \subseteq A_1 \subseteq \dots \subseteq M^{n+k}$.

We define $B(f_{\phi_1}, \dots, f_{\phi_k}) \subseteq (M^{n+k})^\omega$ to consist of chains $(b_i)_{i \in \mathbb{N}}$ that satisfy the following:

- (1) $b_0 \in A_0$
- (2) Suppose $l \in \mathbb{N}$ is such that $A_l \setminus \{b_0, \dots, b_l\} \neq \emptyset$ but $A_j \setminus \{b_0, \dots, b_l\} = \emptyset$ for each $j < l$. If $b_l = (b_l^1, \dots, b_l^{n+k})$ then $b_{l+1} = (b_{l+1}^1, \dots, b_{l+1}^{n+k})$, where
 - (2.a) $b_{l+1}^j \in Col(A_l) \cap (Col\{b_q : q \leq l\} \cup Col(\{b_0\}))$ if $j \in \{p_1, \dots, p_n\}$ and
 - (2.b) $b_{l+1}^j = f_{\phi_s}(b_{l+1}^1, \dots, b_{l+1}^{j-1})$ if $j \in \{m_1, \dots, m_k\}$ and $m_s = j$ (if $j = 1$ then

φ is equivalent to a sentence of the form $\exists x\phi$ with x free in ϕ , so we may interpret $f_{\phi_s}(b_{i+1}^1, \dots, b_{i+1}^{j-1})$ as some $d \in M$ such that $\mathcal{M} \models \phi(d)$.

Claim 2. For each finite A_j , $j \in \mathbb{N}$, and each $(b_i)_{i \in \mathbb{N}} \in B(f_{\phi_1}, \dots, f_{\phi_k}) \subseteq (M^n)^\omega$, there is $m \in \mathbb{N}$ such that

$$A_j \subseteq \{b_i : i \leq m\}.$$

We prove the claim by induction on $l \in \mathbb{N}$. Suppose $m \in \mathbb{N}$ and let $l \in \mathbb{N}$ be the smallest natural number such that $A_l \setminus \{b_0, \dots, b_m\} \neq \emptyset$ but $A_j \setminus \{b_0, \dots, b_m\} = \emptyset$ for each $j < l$. If A_l is infinite then the claim is clear, so we assume A_l is finite. Since $A_i \subseteq A_{i+1}$ we may also assume $l \geq 2$. We show by induction on $j \in \{1, \dots, n+k\}$ that for any $a = (a_1, \dots, a_{n+k}) \in A_l$ there is $q \in \mathbb{N}$ such that $b_q = (b_q^1, \dots, b_q^{n+k})$ and $b_q^p = a_p$ for every $p \leq j$.

(a) $j = 1$: If φ is equivalent to a sentence of the form $\exists x\phi$ with x free in ϕ , then, as was noted above, there is some $s \in M$ such that $a = d = b_i^1$ for any $i \in \mathbb{N}$. Otherwise $a_1^j \in A_{l-1}$ so by the finiteness of A_{l-1} , (2.a) of the of $(b_i)_{i \in \mathbb{N}}$ implies there is $q \in \mathbb{N}$ such that $b_q^1 = a_1$.

(b) $j = t+1$: The induction assumption yields that there is $b_q = (b_q^1, \dots, b_q^{n+k})$ such that $b_q^p = a_p$ for all $p \leq t$. Referring to the definition of notation at (*), if $j \in \{m_1, \dots, m_k\}$ then clause (2.b) of the definition of $(b_i)_{i \in \mathbb{N}}$ applies, and so does the last written sentence of the definition of a Skolem closure. By these two, $j \in \{m_1, \dots, m_k\}$ implies $b_q^j = f_{\phi_s}(b_q^1, \dots, b_q^t) = f_{\phi_s}(a_1, \dots, a_t) = a_j$.

Suppose on the contrary that $j \in \{p_1, \dots, p_n\}$. Now $A_l \setminus \{b_0, \dots, b_q\} = \emptyset$ would complete this induction proof, so we assume $A_l \setminus \{b_0, \dots, b_q\} \neq \emptyset$. Then (2) of the definition of $(b_i)_{i \in \mathbb{N}}$ yields firstly that

$$b_q^j \in \text{Col}(A_l) \cap \text{Col}(\{b_p : p < q\})$$

if $j \in \{p_1, \dots, p_n\}$ and secondly, together with finiteness of A_l , that we may assume $b_q^j = d$ for any $d \in \text{Col}(A_l) \cap \text{Col}\{b_p : p < q\}^{(**)}$. Furthermore, the induction assumption states that for every $c = (c_1, \dots, c_{n+k}) \in A_l$ there is $r \in \mathbb{N}$ such that $b_r = (b_r^1, \dots, b_r^{n+k})$ and $b_r^p = a_p$ for any $p \leq t$, so by the finiteness of A_l there is q^* such that $(A_l)_{\leq t} \subseteq \{b_1, \dots, b_{q^*}\}_{\leq t}$ (see the definition of projection for the definition of the notation \leq_t). Now $a_j \in \text{Col}((A_l)_{\leq t})$, and (2) of the definition of $(b_i)_{i \in \mathbb{N}}$ imply that we may assume q is large enough so that

$$a_j \in (\text{Col}\{b_p : p < q\}).$$

Combining this with (**) lets us assume $b_q^j = a_j$. Hence $b_q^p = a_p$ for all $p \leq j$. This completes the induction proving that for any $j \in \{1, \dots, n+k\}$ and any $a = (a_1, \dots, a_{n+k}) \in A_l$ there is $q \in \mathbb{N}$ such that $b_q = (b_q^1, \dots, b_q^{n+k})$ and $b_q^p = a_p$ for every $p \leq j$.

Therefore we have established that for every $a \in A_l$ there is $i \in \mathbb{N}$ such that $a = b_i$. Now the finiteness of A_l entails that there is $m \in \mathbb{N}$ such that $A_j \subseteq \{b_i : i \leq m\}$. \square Claim 2.

Claim 3. For each $(b_i)_{i \in \mathbb{N}} \in B(f_{\phi_1}, \dots, f_{\phi_k})$ and each $i \in \mathbb{N}$ there is $l \in \mathbb{N}$ such that $b_i \in A_l$.

Proof: It suffices to show that for every $i \in \mathbb{N}$ and every $j \in \{1, \dots, n+k\}$ there is $l \in \mathbb{N}$ and $a \in A_l$ such that $b_i^p = a_p$ for all $p \leq j$, where $b_i = (b_i^1, \dots, b_i^{n+k})$ and $a = (a_1, \dots, a_{n+k})$. We prove this by induction on $i \in \mathbb{N}$, $j \in \{1, \dots, n+k\}$.

(a) $i = 0$, $j \in \{1, \dots, n+k\}$: Directly by the definition of $(b_i)_{i \in \mathbb{N}}$.

(b) $i = q + 1$, $j \in \{p_1, \dots, p_n\}$: Then $b_{i+1}^j \in \text{Col}(A_l) \cap \text{Col}(\{b_q : q < i\})$. The induction assumption and the fact that $A_0 \subseteq A_1 \subseteq \dots$ yield now that there is $l \in \mathbb{N}$ such that $\{b_q : q < i\} \subseteq A_l$ and $b_q^p = a_p$ for some $a = (a_1, \dots, a_{n+k}) \in A_{l+1}$ and every $q < i$, $p \leq n+k$. Therefore there is $(a_1, \dots, a_{j-1}, c_j, \dots, c_{n+k}) \in A_{l+1}$ such that $b_i^p = c_p$ for $p \leq j$.

(c) $i = m + 1$, $j \in \{m_1, \dots, m_k\}$: The induction assumption and the fact that $A_0 \subseteq A_1 \subseteq \dots$ imply that there is $l \in \mathbb{N}$ such that $\{b_q : q \leq i\} \subseteq A_l$ and $b_i^p = a_p$ for some $a = (a_1, \dots, a_{n+k}) \in A_{l+1}$ and every $p \leq j - 1$. Therefore the definitions of $(b_i)_{i \in \mathbb{N}}$ and Skolem closure yield that $a_j = f_{\phi_s}(a_1, \dots, a_{j-1}) = f_{\phi_s}(b_i^1, \dots, b_i^{j-1}) = b_i^j$ for some $m_s = j$. \square Claim 2.

Choose now $(b_i)_{i \in \mathbb{N}} \in B(f_{\phi_1}, \dots, f_{\phi_k})$. We rename $\psi(x_1, \dots, x_{n+k})$ as $\sigma(y, z)$ such that $y = (y_1, \dots, y_n) = (x_i)_{i \in \{p_1, \dots, p_n\}}$ and $z = (z_1, \dots, z_k) = (x_i)_{i \in \{m_1, \dots, m_k\}}$, and we permute each b_i , $i \in \mathbb{N}$, into (a_i, d_i) such that $a_i = (a_i^1, \dots, a_i^n) = (b_i)_{i \in \{p_1, \dots, p_n\}}$ and $d_i = (d_i^1, \dots, d_i^k) = (b_i)_{i \in \{m_1, \dots, m_k\}}$. Based on claim 3, $b_i \in A$ for every $i \in \mathbb{N}$ and some $A \in \text{REC}(\psi, \mathcal{M}, \varphi)$ so lemma 4.10 states that $\mathcal{M} \models \sigma(a_i, d_i)$ for every $i \in \mathbb{N}$.

Further, let $I \subseteq \mathbb{N}$ be maximal such that $0 \in I$ and for any $i \in I$ and any $c \in \{c \in M^k : \sigma(a_{i+1}, c)\}$,

$$\text{Col}(\{c\}) \not\subseteq \text{Col}(\{a_0, \dots, a_i\}).$$

To finalize showing that σ and $(a_i, d_i)_{i \in I}$ are as required, it suffices to prove the below:

Claim 4. I is infinite.

Proof: As we observed before,

$$\overline{(B_0 \times f_{\phi_k} B_0, \varphi, \psi, f_{\phi_1}, \dots, f_{\phi_k})} = \bigcup_{i \in \mathbb{N}} A_i$$

for a chain

$$A_0 \subseteq A_1 \subseteq \dots \subseteq M^{n+k},$$

where each A_{i+1} is constructed from A_i as in the definition of Skolem closures. Since $B_0 \times f_{\phi_k} B_0$ is finite, so is each A_i , $i \in \mathbb{N}$, based on the way they are constructed by finitary functions in the said definition. Therefore we can define a chain $(c_i)_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$ there are $p, q \in \mathbb{N}$ such that

$$A_{i+1} \setminus A_i = \{c_p, \dots, c_q\}.$$

Next permute each c_i , $i \in \mathbb{N}$, into (s_i, t_i) such that $s_i = (s_i^1, \dots, s_i^n) = (c_i)_{i \in \{p_1, \dots, p_n\}}$ and $t_i = (t_i^1, \dots, t_i^k) = (c_i)_{i \in \{m_1, \dots, m_k\}}$. Then we define $J \subseteq \mathbb{N}$ to be maximal such that $0 \in J$

and for any $i \in J$ and any $e \in \{e \in M^k : \sigma(s_{i+1}, e)\}$,

$$Col(\{e\}) \not\subseteq Col(\{s_0, \dots, s_i\}).$$

Now A_0 is finite but, as noted earlier in the proof of this lemma, $REC(\psi, \mathcal{M}, \varphi)$ consists of infinite elements only so also $\bigcup_{i \in \mathbb{N}} A_i$ is infinite.

Suppose J was finite. Considering that we have made no other assumptions on $f_{\phi_1}, \dots, f_{\phi_k}$ than that $f_{\phi_i} \in Skolem(\phi_i, \mathcal{M})$ for each $i \in \{1, \dots, k\}$, we can now assume that

$$f_{\phi_s}(c_{i+1}^1, \dots, c_{i+1}^{j-1}) \subseteq Col(\{s_0, \dots, s_i\})$$

for all $j \in \{m_1, \dots, m_k\}$ such that $m_s = j$ and for all $i \in \mathbb{N}$ except for $i \in J$. Then J being finite,

$$f_{\phi_s}(c_{i+1}^1, \dots, c_{i+1}^{j-1}) \subseteq Col(\{s_0, \dots, s_{max(J)}\})$$

for all $j \in \{m_1, \dots, m_k\}$ such that $m_s = j$ and for all $i \in \mathbb{N}$. Since on the other hand, by the definition of Skolem closures,

$$\{c_i : i \in \mathbb{N}\} \subseteq \{c_0\} \cup Col(\{f_{\phi_s}(c_1^1, \dots, c_i^{j-1}) : j \in \{m_1, \dots, m_k\}, m_s = j, i \in \mathbb{N}\}),$$

also $\{c_i : i \in \mathbb{N}\} = \bigcup_{i \in \mathbb{N}} A_i$ becomes finite, which is a contradiction.

Therefore J is infinite and this claim follows from claim 2. \square Claim 4.

\square

Theorem 4.12. *A consistent L -sentence φ is true in infinite models only, if and only if there is an L -formula $\psi(x_1, \dots, x_n, y_1, \dots, y_n)$ such that for all L -models \mathcal{M} satisfying φ there is $(a_i)_{i \in \mathbb{N}}$, $a_i \in M^n$, such that for all $i \in \mathbb{N}$,*

- (i) $\psi(a_i, a_{i+1})$
- (ii) $\neg\psi(a_{i+1}, a_j)$ for $j \leq i$.

Proof. If the conditions hold then φ is clearly true in infinite models only. For the other direction, let us assume φ is true in infinite models only. Then lemma 4.11 yields that there is an L -formula $\sigma(x'_1, \dots, x'_m, y'_1, \dots, y'_k)$ such that for all L -models \mathcal{M} satisfying φ there is $(b_i, c_i)_{i \in \mathbb{N}}$, $b_i \in M^m, c_i \in M^k$, such that for all $i \in \mathbb{N}$,

- (a) $\sigma(b_i, c_i)$
- (b) for any $d \in \{d \in M^k : \sigma(b_{i+1}, d)\}$,

$$Col(\{d\}) \not\subseteq Col(\{b_0, \dots, b_i\}).$$

Define now $x = (x_1, \dots, x_n) = (x'_1, \dots, x'_m, z_1, \dots, z_k)$, $y = (y_1, \dots, y_n) = (z_1^*, \dots, z_m^*, y'_1, \dots, y'_k)$ and

$$\psi(x, y) := \sigma(x'_1, \dots, x'_m, y'_1, \dots, y'_k) \wedge \bigwedge_{i \in \{1, \dots, k\}} z_i = z_i \wedge \bigwedge_{i \in \{1, \dots, m\}} z_i^* = z_i^*.$$

Further, let $(a_i)_{i \in \mathbb{N}}$ be such that for each $i \in \mathbb{N}$, $a_i = (b_{i+1}, c_i)$. Then ψ and $(a_i)_{i \in \mathbb{N}}$ are clearly as required.

\square

4.2 Island models

In this section we construct a class of models that have the property of consisting of finite equivalence classes, "islands", such that elements in distinct islands have a limited number of possible relationships to one another. We prove that all these "Island models" are pseudo-finite.

Lemma 4.13. *Let \mathcal{M} be an L -model and let $P \in L$ be a binary relation symbol such that $P := (P)^\mathcal{M}$ forms an equivalence relation in M .*

Suppose that for all L -atomic formulas $\psi(x_0, \dots, x_n)$ and for all $0 \leq n' < n$ there are finite sets of L -formulas $\{\sigma_0, \dots, \sigma_s\}$ and $\{\tau_0, \dots, \tau_t\}$, such that

(a) the free variable symbols of $\sigma_0, \dots, \sigma_s$ are within $\{x_0, \dots, x_{n'}\}$ and the ones of $\{\tau_0, \dots, \tau_t\}$ within $\{x_{n'+1}, \dots, x_n\}$

(b) if $\bar{a} \in M^{n'+1}$ and $\bar{b} \in M^{n-n'}$ (from now on, we notate $\bar{a}, \bar{b} \in M$), then there are $l \in \{0, \dots, s\}$ and $k \in \{0, \dots, t\}$ such that $\mathcal{M} \models \sigma_l(\bar{a}) \wedge \tau_k(\bar{b})$ (we'll notate $\sigma_l(\bar{a}) \wedge \tau_k(\bar{b})$)

(c) if $l \in \{0, \dots, s\}$ and $k \in \{0, \dots, t\}$, then $\psi(\bar{a}, \bar{b})$ either for all or for no $\bar{a}, \bar{b} \in M$ such that

$$\sigma_l(\bar{a}) \wedge \tau_k(\bar{b}) \wedge \bigwedge_{\substack{i \in \{0, \dots, n'\} \\ j \in \{0, \dots, n^*\}}} \neg P(a_i, b_j)$$

(let us notate from here on that

$$P(\bar{a}, \bar{b}) := \bigwedge_{\substack{i \in \{0, \dots, n'\} \\ j \in \{0, \dots, n^*\}}} P(a_i, b_j).$$

Then for all L -formulas φ there are corresponding sets of formulas $\{\sigma_{\varphi_0}, \dots, \sigma_{\varphi_s}\}$ and $\{\tau_{\varphi_0}, \dots, \tau_{\varphi_t}\}$ satisfying conditions (1) - (3).

Proof. By induction on structure of the formula $\varphi(x_0, \dots, x_n)$.

1° φ is atomic: Directly by assumption.

2° $\varphi = \neg\psi$: If $\{\sigma_{\psi_0}, \dots, \sigma_{\psi_s}\}$ and $\{\tau_{\psi_0}, \dots, \tau_{\psi_t}\}$ satisfy (a)-(c) for ψ , then also φ satisfies them.

3° $\varphi = \psi \wedge \phi$: Formulas $\sigma_{\psi_k} \wedge \sigma_{\phi_l}$ and $\tau_{\psi_m} \wedge \tau_{\phi_p}$, $k \in \{0, \dots, s_\psi\}$, $l \in \{0, \dots, s_\phi\}$, $m \in \{0, \dots, t_\psi\}$, $p \in \{0, \dots, t_\phi\}$ clearly satisfy (a) ja (b), as by induction assumption we

may assume that formulas σ_{ψ_k} ja σ_{ϕ_l} display the same free variables, and that so do also τ_{ψ_m} ja τ_{ϕ_p} . Condition (c): Let $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in M$ be such that

$$\neg P(\bar{a}, \bar{b}) \wedge (\sigma_{\psi_k} \wedge \sigma_{\phi_l})(\bar{a}) \wedge (\tau_{\psi_m} \wedge \tau_{\phi_p})(\bar{b})$$

and

$$\neg P(\bar{c}, \bar{d}) \wedge (\sigma_{\psi_k} \wedge \sigma_{\phi_l})(\bar{c}) \wedge (\tau_{\psi_m} \wedge \tau_{\phi_p})(\bar{d}).$$

Now

$$\neg P(\bar{a}, \bar{b}) \wedge [\sigma_{\psi_k}(\bar{a}) \wedge \tau_{\psi_m}(\bar{b})] \wedge [\sigma_{\phi_l}(\bar{a}) \wedge \tau_{\phi_p}(\bar{b})]$$

and

$$\neg P(\bar{c}, \bar{d}) \wedge [\sigma_{\psi_k}(\bar{c}) \wedge \tau_{\psi_m}(\bar{d})] \wedge [\sigma_{\phi_l}(\bar{c}) \wedge \tau_{\phi_p}(\bar{d})],$$

so

$$\varphi(\bar{a}, \bar{b}) \text{ joss } \psi(\bar{a}, \bar{b}) \wedge \phi(\bar{a}, \bar{b}) \text{ joss } \psi(\bar{c}, \bar{d}) \wedge \phi(\bar{c}, \bar{d}) \text{ joss } \varphi(\bar{c}, \bar{d}).$$

4° $\varphi = \forall x_p \psi$: Due to symmetry we may assume that x_p is free in ψ and that x_0, \dots, x_n are free in φ . Let $\bar{a} \in M^{n'+1}$ and $\bar{b} \in M^{n-n'}$. Induction assumption offers now σ - and τ -formulas satisfying (a)-(c) ψ with *any* distribution of variables between the σ - and τ -formulas. Suppose that formulas $\{\sigma_{\psi_0}^1, \dots, \sigma_{\psi_{s_1}}^1\}$ and $\{\tau_{\psi_0}^1, \dots, \tau_{\psi_{t_1}}^1\}$ satisfy the conditions by a distribution in which x_p appears in the σ -formulas. Let us then notate that $\sigma_{\psi_j}^1 = \sigma_{\psi_j}^1(\bar{x}_a, x_p)$ and $\tau_{\psi_j}^1 = \tau_{\psi_j}^1(\bar{x}_b, x_p)$. Respectively suppose that $\{\sigma_{\psi_0}^2, \dots, \sigma_{\psi_{s_2}}^2\}$ and $\{\tau_{\psi_0}^2, \dots, \tau_{\psi_{t_2}}^2\}$ fulfill the conditions on distribution in which x_p is a σ -variable. Let us then notate $\sigma_{\psi_j}^1 = \sigma_{\psi_j}^1(\bar{x}_a, x_p)$ and $\tau_{\psi_j}^2 = \tau_{\psi_j}^2(\bar{x}_b, x_p)$. In either case we say that $\sigma_{\psi_j}^1 \wedge \tau_{\psi_k}^1$ or $\sigma_{\psi_j}^2 \wedge \tau_{\psi_k}^2$ is a *positive* combination, if $\psi(\bar{a}, \bar{b}, c)$ when $\sigma_{\psi_j}^1(\bar{a}, c) \wedge \tau_{\psi_k}^1(\bar{b})$ or $\sigma_{\psi_j}^2(\bar{a}) \wedge \tau_{\psi_k}^2(\bar{b}, c)$.

Let us next define some formulas and notations. Let $\bar{a} \in M^{n'+1}$ and $\bar{b} \in M^{n-n'}$ and $l_{\bar{a}, \bar{b}} \in \{0, \dots, s_2\}$ as well as $k_{\bar{a}, \bar{b}} \in \{0, \dots, t_1\}$ be such that $\sigma_{\psi_{l_{\bar{a}, \bar{b}}}}^2(\bar{a}) \wedge \tau_{\psi_{k_{\bar{a}, \bar{b}}}}^1(\bar{b})$.

$$A_{\bar{a}, \bar{b}} = \{i \in \{0, \dots, s_1\} \mid \exists x_p (P(\bar{a}, x_p) \wedge \sigma_{\psi_i}^1(\bar{a}, x_p))\}$$

$$B_{\bar{a}, \bar{b}} = \{j \in \{0, \dots, t_2\} \mid \exists x_p (P(\bar{b}, x_p) \wedge \tau_{\psi_j}^2(\bar{b}, x_p))\}$$

$$C_{\bar{a}, \bar{b}} = \{i \in \{0, \dots, s_1\} \mid \exists x_p \sigma_{\psi_i}^1(\bar{a}, x_p)\}$$

$$D_{\bar{a}, \bar{b}} = \{j \in \{0, \dots, t_2\} \mid \exists x_p \tau_{\psi_j}^2(\bar{b}, x_p)\}$$

$$\begin{aligned}\Sigma_{\bar{a},\bar{b}}(\bar{x}_a, \bar{x}_b) &= \bigwedge_{j \in A_{\bar{a},\bar{b}}} \exists x_p \left(P(\bar{x}_a, x_p) \wedge \sigma_{\psi_i}^1(\bar{x}_a, x_p) \right) \wedge \bigwedge_{j \in B_{\bar{a},\bar{b}}} \exists x_p \left(P(\bar{x}_b, x_p) \wedge \tau_{\psi_j}^2(\bar{x}_b, x_p) \right) \\ &\quad \wedge \sigma_{\psi_{l_{\bar{a},\bar{b}}}}^2(\bar{x}_a) \wedge \tau_{\psi_{k_{\bar{a},\bar{b}}}}^1(\bar{x}_b) \\ &\quad \wedge \forall x_p \left(\left(\bigvee_{i \in C_{\bar{a},\bar{b}}} \sigma_{\psi_i}^1(\bar{x}_a, x_p) \right) \wedge \left(\bigvee_{j \in D_{\bar{a},\bar{b}}} \tau_{\psi_j}^2(\bar{x}_b, x_p) \right) \right)\end{aligned}$$

$$\{\Sigma_{\bar{t},\bar{u}} : \bar{t}, \bar{u} \in M\} = \{\Sigma_k : k \in U\}$$

$$I, J \subseteq U$$

$$\begin{aligned}\sigma_{\psi_I} &= \left(\bigwedge_{l \in I} \exists \bar{x}_b \Sigma_l(\bar{x}_a, \bar{x}_b) \right) \wedge \bigwedge_{l \in U \setminus I} \neg \exists \bar{x}_b \Sigma_l(\bar{x}_a, \bar{x}_b) \\ \tau_{\psi_J} &= \left(\bigwedge_{k \in J} \exists \bar{x}_a \Sigma_k(\bar{x}_a, \bar{x}_b) \right) \wedge \bigwedge_{k \in U \setminus J} \neg \exists \bar{x}_a \Sigma_k(\bar{x}_a, \bar{x}_b).\end{aligned}$$

Let us show that defined as above, there is a finite amount of formulas σ_{ψ_I} ja τ_{ψ_J} , and that they realize (a)-(c) for φ .

Finiteness: This depends on the size of the set $\{\Sigma_k : k \in U\} = \{\Sigma_{\bar{t},\bar{u}} : \bar{t}, \bar{u} \in M\}$. For each $\bar{t}, \bar{u} \in M$ the formula $\Sigma_{\bar{t},\bar{u}}$ consists of predicate P , variables \bar{x}_a and \bar{x}_b , and of finitely many σ - ja τ - formulas for ψ , as well as of sets A, B, C ja D that are subsets of $\{0, \dots, s_1\}$ and $\{0, \dots, t_1\}$. Thus there can be only a finite amount of formulas $\Sigma_{\bar{t},\bar{u}}$, which means there is only a finite amount of formulas σ_{ψ_I} ja τ_{ψ_J} .

(a): Clear, since the only free variable symbols in $\Sigma_{\bar{t},\bar{u}}, \bar{t}, \bar{u} \in M$ are $\bar{x}_a, \bar{x}_b = (x_0, \dots, x_{n'}), (x_{n'+1}, \dots, x_n)$.

(b): Let $\bar{t}, \bar{u} \in M$. According to the induction assumption there is $l \in \{0, \dots, s_2\}$ ja $k \in \{0, \dots, t_1\}$ such that $\sigma_{\psi_l}^2(\bar{t}) \wedge \tau_{\psi_k}^1(\bar{u})$, in addition to which the induction assumption provides $r \in M$ with some $i \in \{0, \dots, s_1\}$ ja $j \in \{0, \dots, t_2\}$ such that $\sigma_{\psi_i}^1(\bar{t}, r) \wedge \tau_{\psi_j}^2(\bar{u}, r)$. Thus at least $\Sigma_{\bar{t},\bar{u}}(\bar{t}, \bar{u})$ implying that $\Sigma_k(\bar{t}, \bar{u})$ for some $k \in U$. Therefore there are non-empty $I, J \subseteq U$ such that $\sigma_{\psi_I}(\bar{t}) \wedge \tau_{\psi_J}(\bar{u})$.

(c): Let $\bar{c}, \bar{d}, \bar{t}, \bar{u} \in M$, $\neg P(\bar{c}, \bar{d})$, $\neg P(\bar{t}, \bar{u})$, $I, J \subseteq U$, $\sigma_{\psi_I}(\bar{t}) \wedge \tau_{\psi_J}(\bar{u})$, $\sigma_{\psi_I}(\bar{c}) \wedge \tau_{\psi_J}(\bar{d})$ and $\varphi(\bar{t}, \bar{u})$. Let us show that $\varphi(\bar{c}, \bar{d})$.

According to condition (b) there is $k \in U$ such that $\Sigma_k(\bar{t}, \bar{u})$. Now must be $k \in I \cup J$, so there is $\bar{q} \in M^{n'+1}$ and $\bar{r} \in M^{n-n'}$ such that $\Sigma_k(\bar{c}, \bar{r})$ ja $\Sigma_k(\bar{q}, \bar{d})$. If now

$$\begin{aligned}\Sigma_k(\bar{x}_a, \bar{x}_b) &= \bigwedge_{j \in A} \exists x_p \left(P(\bar{x}_a, x_p) \wedge \sigma_{\psi_i}^1(\bar{x}_a, x_p) \right) \wedge \bigwedge_{j \in B} \exists x_p \left(P(\bar{x}_b, x_p) \wedge \tau_{\psi_j}^2(\bar{x}_b, x_p) \right) \\ &\quad \sigma_{\psi_l}^2(\bar{x}_a) \wedge \tau_{\psi_k}^1(\bar{x}_b) \wedge \forall x_p \left(\left(\bigvee_{i \in C} \sigma_{\psi_i}^1(\bar{x}_a, x_p) \right) \wedge \left(\bigvee_{j \in D} \tau_{\psi_j}^2(\bar{x}_b, x_p) \right) \right) \\ &=: \phi(\bar{x}_a) \wedge \eta(\bar{x}_b) \wedge \sigma_{\psi_l}^2(\bar{x}_a) \wedge \tau_{\psi_k}^1(\bar{x}_b) \wedge \forall x_p \theta(\bar{x}_a, \bar{x}_b),\end{aligned}$$

then

$$\begin{aligned}\phi(\bar{c}) \wedge \eta(\bar{r}) \wedge \sigma_{\psi_l}^2(\bar{c}) \wedge \tau_{\psi_k}^1(\bar{r}) \\ \wedge \forall x_p \left(\left(\bigvee_{i \in C} \sigma_{\psi_i}^1(\bar{c}, x_p) \right) \wedge \left(\bigvee_{j \in D} \tau_{\psi_j}^2(\bar{r}, x_p) \right) \right)\end{aligned}$$

and

$$\begin{aligned}\phi(\bar{q}) \wedge \eta(\bar{d}) \wedge \sigma_{\psi_l}^2(\bar{q}) \wedge \tau_{\psi_k}^1(\bar{d}) \\ \wedge \forall x_p \left(\left(\bigvee_{i \in C} \sigma_{\psi_i}^1(\bar{q}, x_p) \right) \wedge \left(\bigvee_{j \in D} \tau_{\psi_j}^2(\bar{d}, x_p) \right) \right).\end{aligned}$$

Now for every $h \in M$ there is $i \in C$ and $j \in D$ such that $\sigma_{\psi_i}^1(\bar{c}, h)$ and $\tau_{\psi_j}^2(\bar{d}, h)$, so we obtain

$$\begin{aligned}\phi(\bar{c}) \wedge \eta(\bar{d}) \wedge \sigma_{\psi_l}^2(\bar{c}) \wedge \tau_{\psi_k}^1(\bar{d}) \\ \forall x_p \left(\left(\bigvee_{i \in C} \sigma_{\psi_i}^1(\bar{c}, x_p) \right) \wedge \left(\bigvee_{j \in D} \tau_{\psi_j}^2(\bar{d}, x_p) \right) \right)\end{aligned}$$

that is equivalent to $\Sigma_k(\bar{c}, \bar{d})$.

Because $\Sigma_k(\bar{t}, \bar{u})$, we have

$$\begin{aligned}\bigwedge_{j \in A} \exists x_p \left(P(\bar{t}, x_p) \wedge \sigma_{\psi_i}^1(\bar{t}, x_p) \right) \wedge \bigwedge_{j \in B} \exists x_p \left(P(\bar{u}, x_p) \wedge \tau_{\psi_j}^2(\bar{u}, x_p) \right) \\ \sigma_{\psi_l}^2(\bar{t}) \wedge \tau_{\psi_k}^1(\bar{u}) \wedge \forall x_p \left(\left(\bigvee_{i \in C} \sigma_{\psi_i}^1(\bar{t}, x_p) \right) \wedge \left(\bigvee_{j \in D} \tau_{\psi_j}^2(\bar{u}, x_p) \right) \right).\end{aligned}$$

If $h \in M$ then there is $i \in C$ ja $j \in D$, such that $\sigma_{\psi_i}^1(\bar{t}, h) \wedge \tau_{\psi_k}^1(\bar{u})$ and $\sigma_{\psi_l}^2(\bar{t}) \wedge \tau_{\psi_j}^2(\bar{u}, h)$. Let now $\bar{a}, \bar{b} \in M$ be such that $\Sigma_k = \Sigma_{\bar{a}, \bar{b}}$. Then there are three option:

i) $\neg P(h, \bar{a}) \wedge \neg P(h, \bar{b})$: Now $\sigma_{\psi_l}^2(\bar{a}) \wedge \tau_{\psi_k}^1(\bar{b})$ and for some $n \in C_{\bar{a}, \bar{b}}$, $m \in D_{\bar{a}, \bar{b}}$ holds $\sigma_{\psi_n}^1(\bar{t}, h) \wedge \tau_{\psi_m}^2(\bar{u}, h)$. In addition $\neg P(h, \bar{t})$ or $\neg P(h, \bar{u})$ and $\psi(\bar{t}, h, \bar{u})$, so at least $\sigma_{\psi_n}^1 \wedge \tau_{\psi_k}^1$ tai $\sigma_{\psi_l}^2 \wedge \tau_{\psi_m}^2$ is a positive combination. Now both $\neg P(h, \bar{a})$ and $\neg P(h, \bar{b})$, so $\psi(\bar{a}, h, \bar{b})$.

ii) $P(h, \bar{a})$: Now $\tau_{\psi_k}^1(\bar{b})$, $\tau_{\psi_k}^1(\bar{u})$, for some $i' \in A_{\bar{a}, \bar{b}}$ holds $\sigma_{\psi_{i'}}^1(\bar{a}, h)$, and there is $h' \in M$, $P(h', \bar{t})$ such that $\sigma_{\psi_{i'}}^1(\bar{t}, h')$. Then $\neg P(h', \bar{u})$ ja $\psi(\bar{t}, h', \bar{u})$, so $\sigma_{\psi_{i'}}^1 \wedge \tau_{\psi_k}^1$ is a positive combination; since $\neg P(h, \bar{b})$, then also $\psi(\bar{a}, h, \bar{b})$.

iii) $P(h, \bar{b})$: By symmetry we can see that $\psi(\bar{a}, h, \bar{b})$ the same way as in ii).

Thus we see that all the $\sigma_{\psi_i}^1 \wedge \tau_{\psi_j}^1$ and $\sigma_{\psi_i}^2 \wedge \tau_{\psi_j}^2$ - combinations appearing in $\Sigma_{\bar{a}, \bar{b}}$ are positive. Since $\Sigma_k(\bar{c}, \bar{d})$, we obtain

$$\sigma_{\psi_l}^2(\bar{c}) \wedge \tau_{\psi_k}^1(\bar{d}) \wedge \forall x_p \left(\left(\bigvee_{i \in C} \sigma_{\psi_i}^1(\bar{c}, x_p) \right) \wedge \left(\bigvee_{j \in D} \tau_{\psi_j}^2(\bar{d}, x_p) \right) \right),$$

meaning that for every $h \in M$ there is $i \in C$ and $j \in D$ such that $\sigma_{\psi_i}^1(\bar{c}, h) \wedge \tau_{\psi_k}^1(\bar{d})$ and $\sigma_{\psi_l}^2(\bar{c}) \wedge \tau_{\psi_j}^2(\bar{d}, h)$. Now either $\neg P(h, \bar{c})$ or $\neg P(h, \bar{d})$, so $\psi(\bar{c}, h, \bar{d})$ for every $h \in M$, so we obtain $\varphi(\bar{c}, \bar{d})$. Therefore by symmetry, $\varphi(\bar{c}, \bar{d})$ if and only if $\varphi(\bar{t}, \bar{u})$.

□

Definition 4.14. Let L be an alphabet that includes an infinite number of function symbols, let $P \notin L$ a new binary relation symbol and let \mathcal{M} be an L -model. We say that \mathcal{M} is an **island model**, if there is an $L \cup \{P\}$ -model \mathcal{M}^* satisfying the assumptions of lemma 4.13 such that $\mathcal{M}^* \upharpoonright L = \mathcal{M}$ and every finite set of equivalence relations of $P^{\mathcal{M}^*}$ combined with interpretations induced by \mathcal{M}^* is a finite submodel of \mathcal{M}^* . We call such an $\mathcal{N} \subseteq \mathcal{M}^*$ fulfilling these conditions an island submodel of \mathcal{M}^* and we notate $\mathcal{N} \subset_{\text{island}} \mathcal{M}^*$. Furthermore, if $\mathcal{N} \subset_{\text{island}} \mathcal{H}$ ja $\mathcal{H} \subset_{\text{island}} \mathcal{M}^*$, we notate $\mathcal{N} \subset_{\text{island}} \mathcal{M}$.

Theorem 4.15. Let \mathcal{M} be an L -island model and let L and \mathcal{M}^* be as in definition 4.14. Then for all such L -formulas φ that $\mathcal{M}^* \models \varphi$ there is $\mathcal{N} \subset_{\text{island}} \mathcal{M}^*$, such that if $\mathcal{N} \subset_{\text{island}} \mathcal{H} \subset_{\text{island}} \mathcal{M}^*$, then $\mathcal{H} \models \varphi$. Especially \mathcal{M}^* , and thus also \mathcal{M} , are pseudo-finite.

Proof. Based on lemma 4.13, it suffices to prove the claim for L -formulas in the quantifiers-first-form. We use induction on the structure of an L -formula in the said form.

Let us form the induction assumption as follows: For each formula $\psi(x_0, \dots, x_n)$ and interpretation function s of model \mathcal{M}^* there is $\mathcal{N}_{\psi, s} \subset_{\text{island}} \mathcal{M}^*$ such that if $\mathcal{N}_{\psi, \bar{a}} \subset_{\text{island}}$

$\mathcal{K} \subset_{\text{island}} \mathcal{M}^*$, $a_0, \dots, a_n \in \mathcal{M}^*$ and $\mathcal{M}^* \models \psi(a_0, \dots, a_n)$, then $\mathcal{H} \models \psi(a_0, \dots, a_n)$, where $\mathcal{H} \subset_{\text{island}} \mathcal{M}^*$ and

$$H = \text{dom}(\mathcal{H}) = \text{dom}(\mathcal{K}) \cup \{b \in \mathcal{M}^* : P(b, a_i) \text{ for all } i \in \{0, \dots, n\}\}.$$

1° $\varphi(x_0, \dots, x_n)$ is unquantified: Let s be an interpretation function in \mathcal{M}^* , let $a_0, \dots, a_n \in \mathcal{M}^*$, let $\mathcal{M}^* \models \varphi(a_0, \dots, a_n)$, and let

$$N_{\varphi, s} = \{b \in M^* \mid P(b, s(t)), t \text{ is a term displaying in formula } \varphi\},$$

$$\mathcal{N}_{\varphi, s} \subset_{\text{island}} \mathcal{K} \subset_{\text{island}} \mathcal{M}^* \text{ and}$$

$$H = K \cup \{b \in M^* \mid P(b, a_i) \text{ for every } i \in \{0, \dots, n\}\}.$$

Then $\mathcal{H} \models \varphi(a_0, \dots, a_n)$ since $\mathcal{H} \subseteq \mathcal{M}^*$ by the island model's definition.

2° $\varphi = \exists x_p \psi$: According to lemma 4.13 there are formulas $\{\sigma_{\phi_0}^{n'}, \dots, \sigma_{\phi_s}^{n'}\}$ ja $\{\tau_{\phi_0}^{n'}, \dots, \tau_{\phi_t}^{n'}\}$ that satisfy the lemma's assumptions for model \mathcal{M}^* , for all L -formulas $\phi(x_0, \dots, x_n)$ and for all the ways of distributing the $n + 1$ variable symbols of ψ between the σ and τ formulas. Let us name sequences $\bar{a}_{\phi_l}^{n'} \in (M^*)^{n'+1}$ and $\bar{b}_{\phi_k}^{n'} \in (M^*)^{n-n'}$ for each $\sigma_{\phi_l}^{n'}$ ja $\tau_{\phi_k}^{n'}$, such that $\sigma_{\phi_l}(\bar{a}_{\phi_l}^{n'})$ and $\tau_{\phi_k}(\bar{b}_{\phi_k}^{n'})$, assuming that such exist. Let $A_{\phi}^{n'}$ and $B_{\phi}^{n'}$ be the finite sets of these (finite) sequences.

If x_p is not free in ψ then $\vdash \psi \leftrightarrow \varphi$, which means the induction assumption holds trivially. Hence we may assume that $\psi = \psi(x_0, \dots, x_n, x_p)$ and $p > n$. Let then $\{\sigma_{\psi_0}^n, \dots, \sigma_{\psi_s}^n\}$ and $\{\tau_{\psi_0}^n, \dots, \tau_{\psi_t}^n\}$ be formulas realizing lemma 4.13 for ψ such that only x_p appears in the τ -formulas.

Let now s be an interpretation function in \mathcal{M}^* , let $\bar{a} = a_0, \dots, a_n \in M^*$ and $e \in M^*$ be such that $\mathcal{M}^* \models \psi(\bar{a}, e)$. Then based on lemma 4.13, either $P(a_i, e)$ for all $i \in \{0, \dots, n\}$ or there is $d \in B_{\psi}^{n'}$ such that $\psi(\bar{a}, d)$ (*).

Let us therefore study $\mathcal{N}_{\varphi, s} \subset_{\text{island}} \mathcal{M}^*$, the domain of which is

$$N_{\psi, s} \cup \{c \in M^* \mid P(c, d) \text{ for some } d \in B_{\psi}^{n'}\},$$

where $N_{\psi, s} = \text{dom}(\mathcal{N}_{\psi, s})$ and $\mathcal{N}_{\psi, s}$ is the island submodel of \mathcal{M}^* satisfying the induction assumption for the formula ψ and the interpretation function s .

Let now $\mathcal{N}_{\varphi, s} \subset_{\text{island}} \mathcal{K} \subset_{\text{island}} \mathcal{M}^*$ and let $\mathcal{H} \subset_{\text{island}} \mathcal{M}^*$ be obtained from model \mathcal{K} by adding to $\text{dom}(\mathcal{K})$ the following set

$$\{b \in \mathcal{M}^* : P(b, a_i) \text{ for every } i \in \{0, \dots, n\}\}.$$

Suppose then $\mathcal{M}^* \models \varphi(\bar{a})$. Now there is $c \in M^*$ such that $\mathcal{M}^* \models \psi(\bar{a}, c)$, so by (*), $\mathcal{M}^* \models \psi(\bar{a}, c)$ for some $c \in M^*$ such that either $c \in B_{\psi}^{n'}$, or $P(\bar{a}, c)$ (meaning $P(a_i, c)$ for all $i \in \{0, \dots, n\}$); in any case $c \in H$. Hence the induction assumption implies $\mathcal{H} \models \psi(\bar{a}, c)$ for some $c \in H$, and so $\mathcal{H} \models \varphi(\bar{a})$. Therefore $\mathcal{N}_{\varphi, s}$ satisfies the induction claim.

3° $\varphi = \forall x_p \psi$: Let $\varphi = \varphi(x_0, \dots, x_n)$ and suppose $\bar{a} \in (M^*)^n$ is such that $\mathcal{M}^* \models \varphi(\bar{a})$. Further, let $\mathcal{N}_{\psi, s} \subset_{\text{island}} \mathcal{K} \subset_{\text{island}} \mathcal{M}^*$, $K = \text{dom}(\mathcal{K})$ and $e \in K \cup \{b \in M^* \mid P(\bar{a}, b)\}$. Then $\mathcal{M}^* \models \psi(\bar{a}, e)$, so according to the induction assumption: If $\mathcal{H} \subset_{\text{island}} \mathcal{M}^*$ is such that

$$H = K \cup \{b \in M^* \mid P(\bar{a}, e; b)\},$$

then $\mathcal{H} \models \psi(\bar{a}, e)$. Now either $e \in K$ or $e \in \{b \in M^* \mid P(\bar{a}, b)\}$. In the former case $H = K$ and in the latter, $H = K \cup \{b \in M^* \mid P(\bar{a}, b)\}$. However, defining \mathcal{K}' and \mathcal{H}' otherwise as \mathcal{K} and \mathcal{H} were but with the difference that $K' = K \cup \{b \in M^* \mid P(\bar{a}, b)\}$, implies that $H' = K' \cup \{b \in M^* \mid P(\bar{a}, b)\}$ in any case but, by the induction assumption, $\mathcal{H}' \models \psi(\bar{a}, e)$ still. Note that \mathcal{H}' is the same undepending on the $e \in K \cup \{b \in M^* \mid P(\bar{a}, b)\} = H'$ chosen and $\mathcal{H}' \models \psi(\bar{a}, e)$ at any rate. Thus define

$$\mathcal{N}_{\varphi, s} = \mathcal{N}_{\psi, s}$$

and let $\mathcal{N}_{\varphi, s} \subset_{\text{island}} \mathcal{K}^* \subset_{\text{island}} \mathcal{M}^*$ and $\mathcal{H}^* \subset_{\text{island}} \mathcal{M}^*$ where

$$H^* = K^* \cup \{b \in M^* \mid P(\bar{a}, b)\}.$$

Now $\mathcal{H}^* = \mathcal{H}'$ so $\mathcal{H}^* \models \psi(\bar{a}, e)$ for any $e \in H^*$. Thus $\mathcal{H}^* \models \varphi(\bar{a})$, so $\mathcal{N}_{\varphi, s}$ is as required. \square

Remark 4.16. The theorem does not generalize to the form that it suffices for the islands, the equivalence classes of relation P , to be pseudo-finite. We develop a counter example below.

Example 4.17. *There is a non-pseudofinite model that otherwise fulfills the definition of an island model but has an infinite, yet pseudofinite, "island".*

Proof. Let \mathcal{V} be pseudo-finite $\{R, V\}$ -model, where V is a unary relation symbol such that $V^{\mathcal{V}} = V$. Then let $b \notin V$ and let \mathcal{W} be another $\{R, V\}$ -model such that $W = V \cup \{b\}$ and $\mathcal{W} \upharpoonright V = \mathcal{V}$. Suppose R is a ternary relation symbol and $V = \{a_i : i \in \mathbb{N}\}$. Let $\mathcal{W} \models R(a_i, b, a_j)$ and let $\mathcal{W} \models R(a_i, a_j, b)$ and $\mathcal{W} \models R(b, a_i, a_j)$ be true precisely when $i \leq j$, and let $R(a, b, b)$ be $R(b, b, a)$ false in \mathcal{W} for all $a \in W$. Then the following is true in \mathcal{W} :

$$\begin{aligned} & \exists v \left(\forall x \forall y \forall z \left((x \neq v \rightarrow R(x, v, x)) \wedge (R(x, v, y) \wedge R(y, v, z) \rightarrow R(x, v, z)) \right) \right) \\ & \wedge \exists x_0 \left(\forall x_1 \left(R(x_0, v, x_1) \rightarrow \exists y \left(R(x_1, v, y) \wedge \forall z (R(y, v, z) \rightarrow \neg R(z, v, x_1)) \right) \right) \right). \end{aligned}$$

Now \mathcal{W} is not pseudofinite, but every $\{R, V\}$ -atomic formula, i.e. formulas of the form $x = y$ or $R(x, y, z)$, satisfy the conditions of lemma 4.13, as we can expand model \mathcal{W} $\{R, V, P\}$ to \mathcal{W}^* by adding an island separating equivalence relation $P^{\mathcal{W}^*} := \{(x, y) \in W^2 : x, y \in V \text{ or } x = y = v\}$, and define the σ - and τ -formulas for $R(x, y, z)$ as follows (due to symmetry we may assume that x and y are σ -variables and z is a τ -variable):

$$\tau_0 = \neg V(z)$$

$$\tau_1 = V(z)$$

$$\sigma_0 = V(x) \wedge V(y) \wedge \forall z(\neg V(z) \rightarrow R(x, y, z))$$

$$\sigma_1 = V(x) \wedge V(y) \wedge \forall z(\neg V(z) \rightarrow \neg R(x, y, z))$$

$$\sigma_2 = \neg V(y) \vee \neg V(x).$$

Now for all $t, u, w \in W$ there are i and j such that $\sigma_i(t, u) \wedge \tau_j(w)$, and if $P(t, u)$ ja $\neg P(u, w)$, then either

(1) $V(t) \wedge V(u)$ in which case $\tau_0(w)$ and $\sigma_0(t, u)$ (implying $R(t, u, w)$) or $\sigma_1(t, u)$, (implying $\neg R(t, u, w)$)

or

(2) $\neg V(t) \vee \neg V(u)$, in which case $\sigma_2(t, u)$ and $\tau_1(w)$, implying $\neg R(t, u, w)$; $\tau_0(w)$ is not possible now since $\neg P(t, w) \wedge \neg P(u, w)$.

Thus \mathcal{W} otherwise satisfies the definition of an island model except that it has an infinite "island" (meaning an equivalence class of P). However, \mathcal{W} is not pseudo-finite.

□

Example 4.18. *A model with empty alphabet.*

Let $\mathcal{M} = (M)$. Then $\mathcal{M}^* = (M; P)$ satisfies the conditions in lemma 4.13, when P is a binary relation symbol, $P^{\mathcal{M}^*}(a, b)$ iff $a = b$ and $\sigma_{x_0=x_1} = \tau_{x_0=x_1} = \forall x_0(x_0 = x_0)$. In addition \mathcal{M} is an island model because \mathcal{M}^* fulfills definition 4.14. Thus theorem 4.15 yields \mathcal{M} pseudo-finite.

Example 4.19. *Suppose \mathcal{A}_i , $i \in I$, are models of empty alphabet and $\mathcal{B} = (\bigcup_{i \in I} A_i; P_i : i \in I)$, where $P_i : i \in I$ are binary relation symbols such that $P_i(a, b)$ iff $a, b \in A_i$ for*

each $i \in I$. Suppose R is an extra binary relation symbol and $\{R, P_i : i \in I\}$ -model \mathcal{B}^* an expansion of model \mathcal{B} , such that $\mathcal{B}^* \upharpoonright (P_i : i \in I) = \mathcal{B}$ and $R^{\mathcal{B}^*}(a, b)$ iff $a = b$. Then \mathcal{B} realizes the conditions of an island model and is therefore pseudofinite, which we see by defining the equivalence classes and formulas required by the following way: $P = R$, $\sigma_{x_0=x_1} = \tau_{x_0=x_1} = \forall x_0(x_0 = x_0)$, $\sigma_{P_i,0} = \exists x_1 P_i(x_0, x_1)$, $\tau_{P_i,0} = \exists x_0 P_i(x_0, x_1)$, $\sigma_{P_i,1} = \neg \exists x_1 P_i(x_0, x_1)$ and $\tau_{P_i,1} = \neg \exists x_0 P_i(x_0, x_1)$.

Example 4.20. Suppose $\{P_i^j : i \in I\}$ -model \mathcal{B}_j is like \mathcal{B} of the previous example, for each $j \in J$, and suppose $\mathcal{C} = (\bigcup_{j \in J} B_j; P'_j : j \in J)$, where $P'_j : j \in J$ are binary relation symbols such that $P'_j(a, b)$ iff $a, b \in B_j$ for each $j \in J$. Then model \mathcal{C} can be found pseudo-finite by a respective technique as with \mathcal{B} in the previous example.

Example 4.21. "Island version" of the standard model of number theory

Let us define

$$\mathcal{N} = \left(\bigcup_{i \in \mathbb{N}} \mathbb{Z}_i; +, \times, \mathbf{0}_i, \mathbf{1}_i : i \in \mathbb{N} \right)$$

as follows: for each $i \in \mathbb{N}$, the set of of congruence classes, $\mathbb{Z}_{i+1} = \{0_i, \dots, i_i\}$, $\mathbf{0}_i^{\mathcal{N}} = 0_i$, $\mathbf{1}_i^{\mathcal{N}} = 1_i$, and the function symbols $+$ and \times are interpreted as normal within the congruence classes, but if $n, m \leq i$ and $j < i$, then $n_i + m_j = n_i \times m_j = 0_0$. Now \mathcal{N} is an island model, when we interpret $P^{\mathcal{N}}(n_i, m_j)$ iff $i = j$, since for all $i \neq j$, $n_i + m_j = p_k$ and $n_i \times m_j = p_k$ hold exactly if $p_k = \mathbf{0}_0$, and for instance, $n_i + p_k = m_j$ holds precisely if $m_j = \mathbf{0}_0$.

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